Introduction to Stochastic Calculus

Math 545 - Duke University

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Contents

CHAPTER 1

Introduction

1. Motivations

Evolutions in time with random influences/random dynamics. Let $N(t)$ be the "number of rabbits in some population" or "the price of a stock". Then one might want to make a model of the dynamics which includes "random influences". A (very) simple example is

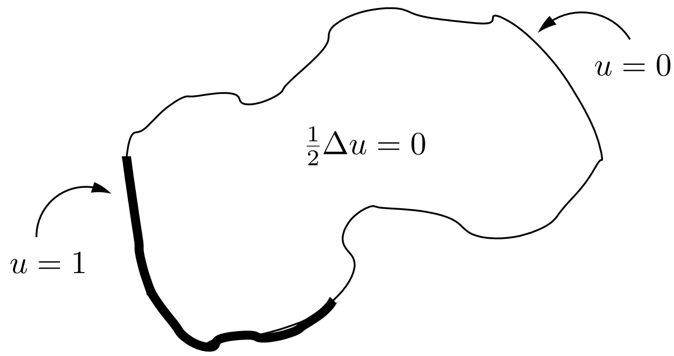
$$\frac{dN(t)}{\mathrm{d}t} = a(t)N(t) \text{ where } a(t) = r(t) + \text{ "noise" }. \tag{1.1}$$

Making sense of "noise" and learning how to make calculations with it is one of the principal objectives of this course. This will allow us predict, in a probabilistic sense, the behavior of $N(t)$. Examples of situations like the one introduced above are ubiquitous in nature:

i) **The gambler's ruin problem** We play the following game: We start with 3$ in our pocket and we flip a coin. If the result is tail we loose one dollar, while if the result is positive we win one dollar. We stop when we have no money to bargain, or when we reach 9$. We may ask: what is the probability that I end up broke?
ii) **Population dynamics/Infectious diseases** As anticipated, (1.1) can be used to model the evolution in the number of rabbits in some population. Similar models are used to model the number of genetic mutations an animal species. We may also think about $N(t)$ as the number of sick individuals in a population. Reasonable and widely applied models for the spread of infectious diseases are obtained by modifying (1.1), and observing its behavior. In all these cases, one may be interested in knowing if it is likely for the disease/mutation to take over the population, or rather to go extinct.
iii) **Stock prices** We may think about a set of M risky investments (e.g. a stock), where the price $N_i(t)$ for $i \in \{1, \dots M\}$ per unit at time t evolves according to (1.1). In this case, one one would like to optimize his/her choice of stocks to maximize the total value $\sum_{i=1}^{M} \alpha_i N_i(t)$ at a later time T.

Connections with diffusion theory and PDEs. There exists a deep connection between noisy processes such as the one introduced above and the deterministic theory of partial differential equations. This starling connection will be explored and expanded upon during the course, but we anticipate some examples below:

i) **Dirichlet problem** Let $u(x)$ be the solution to the PDE given below with the noted boundary conditions. Here $\Delta = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The amazing fact is the following: If we start a Brownian motion diffusing from a point (x_0, y_0) inside the domain then the probability that it first hits the boundary in the darker region is given by $u(x_0, y_0)$.
ii) **Black Scholes Equation** Suppose that at time $t = 0$ the person in *iii)* is offered the right (without obligation) to buy one unit of the risky asset at a specified price S and at a specified future time $t = T$. Such a right is called a *European call option*. How much should the person be willing to pay for such an option? This question can be answered by solving the famous Black Scholes equation, giving for any stock price $N(t)$ the right value S of the European option.



2. Outline For a Course

What follows is a rough outline of the class, giving a good indication of the topics to be covered, though there will be modifications.

i) Weeks 1-2: Motivation and Introduction to Stochastic Process
   (a) Motivating Examples: Random Walks, Population Model with noise, Black-Scholes, Dirichlet problems
   (b) Themes: Direct calculation with stochastic calculus, connections with PDEs
   (c) Introduction: Probability Spaces, Expectations, σ-algebras, Conditional expectations, Random walks and discrete time stochastic processes. Continuous time stochastic processes and characterization of the law of a process by its finite dimensional distributions (Kolmogorov Extension Theorem). Markov Process and Martingales.

ii) Weeks 3-4: Brownian motion and its Properties
   (a) Definitions of Brownian motion (BM) as a continuous Gaussian process with independent increments. Chapman-Kolmogorov equation, forward and backward Kolmogorov equations for BM. Continuity of sample paths (Kolmogorov Continuity Theorem). BM and more Markov process and Martingales.
   (b) First and second variation (a.k.a variation and quadratic variation) Application to BM

iii) Week 5: Stochastic Integrals
   (a) The Riemann-Stieltjes integral. Why can't we use it ?
   (b) Building the Itô and Stratonovich integrals (Making sense of "$\int_0^t \sigma \, \mathrm{d}B$.")
   (c) Standard properties of integrals hold: linearity, additivity
   (d) Itô isometry: $\mathbb{E}(\int f \, \mathrm{d}B)^2 = \mathbb{E} \int f^2 \, \mathrm{d}s$.

iv) Week 6: Itô's Formula and Applications
   (a) Change of variable
   (b) Connections with PDEs and the Backward Kolmogorov equation

v) Week 7: Stochastic Differential Equations
   (a) What does it mean to solve an SDE ?
   (b) Existence of solutions (Picard iteration), Uniqueness of solutions

vi) Week 8-9: Stopping Times
   (a) Definition. σ-algebra associated to stopping time. Bounded stopping times. Doob's optional stopping theorem
   (b) Dirichlet Problems and hitting probabilities
   (c) Localization via stopping times

vii) Week 10: Levy-Doob theorem and Girsonov's Theorem
   (a) How to tell when a continuous martingale is a Brownian motion
   (b) Random time changes to turn a Martingale into a Brownian motion

(c) Hermite Polynomials and the exponential martingale

(d) Girsanov's Theorem, Cameron-Martin formula, and changes of measure

    (1) The simple example of i.i.d Gaussian random variables shifted
    (2) Idea of Importance sampling and how to sample from tails
    (3) The shift of a Brownian motion
    (4) Changing the drift in a diffusion

viii) Week 11: Feller Theory of one dimensional diffusions

(a) Speed measures, natural scales, the classification of boundary point.

ix) Week 12-13: Applications

(a) Option Pricing and the Black-Scholes equation

(b) Population biology and Chemical Kinetics

(c) Stochastic Control, Signal Processing and Reinforcement learing

CHAPTER 2

Probabilistic Background

1. Countable probability spaces

EXAMPLE 1.1. *We begin with the following motivating example. Consider a random sequence* $\omega = \{\omega_i\}_{i=0}^N$ *where*

$$\omega_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

independent of the other ω_j*'s. We will also write this as*

$$\mathbb{P}[(\omega_1, \omega_2, \dots, \omega_N) = (s_1, s_2, \dots, s_N)] = p^{n_+}(1-p)^{N-n_+}$$

for $s_i = \pm 1$*, where* $n_+ := |\{i \ : \ s_i = +1\}|$*. We can group the possible outcomes with* $\omega_1 = +1$*:*

$$A_1 = \{\omega \in \Omega \ : \ \omega_1 = 1\}\,.$$

and compute the probability of such an event:

$$\mathbb{P}[A_1] = \sum_{\omega \in A_1} \mathbb{P}[\omega] = p.$$

Let Ω *be the set of all such sequences of length* N *(i.e.* $\Omega = \{-1,1\}^N$ *), and consider now the sequence of functions* $\{X_n : \Omega \to \mathbb{Z}\}$ *where*

$$\begin{aligned} X_0(\omega) &= 0 \qquad (2.1) \\ X_n(\omega) &= \sum_{i=1}^n \omega_i \end{aligned}$$

for $n \in \{1, \cdots, N\}$*. This sequence is a* random walk *of length* N *(a simple example of a stochastic process) and we can compute its expectation:*

$$\mathbb{E}[X_2] = \sum_{i \in \{-2,0,2\}} i\mathbb{P}[X_2 = i] = 2p^2 - 2(1-p)^2 = 2(2p-1)\,.$$

This expectation changes if we assume that we have some information *on the state of the random walk at an earlier time:*

$$\mathbb{E}[X_2 | X_1 = 1] = \sum_{i \in \{-2,0,2\}} i\mathbb{P}[X_2 = i | X_1 = 1] = 2p + 0\,(1-p) = 2p\,.$$

We now recall some basic definitions from the theory of probability which will allow us to put this example on solid ground.

In the above example, the set Ω is called the *sample space* (or outcome space). Intuitively, each $\omega \in \Omega$ is a possible outcome of all of the randomness in our system. The subsets of Ω (the sets of outcomes we want to compute the probability of) are referred to as the *events* and the measure given by $\mathbb{P}$ on subsets $A \subseteq \Omega$ is called the *probability measure*, giving the chance of the various outcomes. Finally, each X_n is an example of an integer-valued *random variable.* We will refer to this collection of random variables as *random walk.*

In the above setting where the outcome space Ω consists of a finite number of elements, we are able to define everything in a straightforward way. We begin with a quick recalling of a number of definitions in the countably infinite (possibly finite) setting.

If Ω is countable it is enough to define the probability of each element in Ω. That is to say give a function $p\colon \Omega \to [0,1]$ with $\sum_{\omega\in\Omega} p(\omega) = 1$ and define

$$\mathbb{P}[\omega] = p(\omega)$$

for each $\omega \in \Omega$. An event A is just a subset of Ω. We naturally extend the definition of $\mathbb{P}$ to an event A by

$$\mathbb{P}[A] := \sum_{\omega\in A} P[\omega]\,.$$

Observe that this definition has a number of consequences. In particular, if A_i are disjoint events, that is to say $A_i \subset \Omega$ and $A_i \cap A_k = \varnothing$ if $i \neq j$ then

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$$

and if $A^c := \{\omega \in \Omega \text{ with } \omega \notin A\}$ is the compliment of A then $\mathbb{P}[A] = 1 - \mathbb{P}[A^c]$.

Given two event A and B, the *conditional probability* of A given B is defined by

$$\mathbb{P}[A|B] := \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[B]} \tag{2.2}$$

For fixed B, this is just a new probability measure $\mathbb{P}[\,\cdot\,|B]$ on Ω which gives probability $\mathbb{P}[\omega|B]$ to the outcome $\omega \in \Omega$.

A random variable taking values in some set $\mathbf{X}$ is a function $X\colon \Omega \to \mathbf{X}$. In particular a real-valued *random variable* X is simply a real-valued function $X\ :\ \Omega \to \mathbb{R}$. Throughout this course we will almost exclusively consider real-valued random variables. We can then define the *expected value* of a random variable X (or simply the expectation of X) as

$$\mathbb{E}[X] := \sum_{x\in\mathrm{Range}(X)} x\mathbb{P}[X = x] = \sum_{\omega\in\Omega} X(\omega)\mathbb{P}[\omega]$$

Here we have used the convention that $\{X = x\}$ is short hand for $\{\omega \in \Omega : X(\omega) = x\}$ and the definition of $\mathrm{Range}(X) = \{x \in \mathbf{X} : \exists\omega, \text{with } X(\omega) = x\} = X^{-1}(\Omega)$. We can further define the *covariance* of two random variables X, Y in the same space as $\mathrm{Cov}\,[X, Y] = \mathbb{E}\left[(X - \mathbb{E}\,[X]) \cdot (Y - \mathbb{E}\,[Y])\right]$ and

$$\mathrm{Var}\,[X] := \mathrm{Cov}\,[X, X] = \mathbb{E}\left[X^2 - \mathbb{E}\,[X]^2\right]\,.$$

Two events A and B are *independent* if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. Two random variable are independent if $\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x]\mathbb{P}[Y = y]$. Of course this implies that for any events A and B that $\mathbb{P}[X \in A, Y \in B] = \mathbb{P}[X \in A]\mathbb{P}[Y \in B]$ and that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ and that $\mathrm{Cov}\,[X, Y] = 0$. A collection of events A_i is said to be *mutually independent* if

$$\mathbb{P}[A_1 \cap \cdots \cap A_n] = \prod_{i=1}^{n} \mathbb{P}[A_i]\,.$$

Similarly a collection of random variable X_i are mutually independent if for any collection of sets from their range A_i one has that the collection of events $\{X_i \in A_i\}$ are mutually independent. As before, as a consequence one has that

$$\mathbb{E}[X_1 \ldots X_n] = \prod_{i=1}^{n} \mathbb{E}[X_i]\,.$$

Given two **X**-valued random variables Y and Z, for any $z \in \text{Range(Z)}$ we define the *conditional expectation* of Y given $\{Z = z\}$ as

$$\mathbb{E}[Y|Z = z] := \sum_{y \in \text{Range}(Y)} y\mathbb{P}[Y = y|Z = z] \tag{2.3}$$

Which is to say that $\mathbb{E}[Y|Z = z]$ is just the expected value of Y under the probability measure which is given by $P[\,\cdot\,|Z = z]$.

In general, for any event A we can define the conditional expectation of Y given A as

$$\mathbb{E}[Y|A] := \sum_{y \in \text{Range}(Y)} y\mathbb{P}[Y = y \mid A] \tag{2.4}$$

We can extend the definition $\mathbb{E}[Y|Z = z]$ to $\mathbb{E}[Y|Z]$ which we understand to be a function of Z which takes the value $\mathbb{E}[Y|Z = z]$ when $Z = z$. More formally $\mathbb{E}[Y|Z] := h(Z)$ where $h\colon \text{Range}(Z) \to \mathbf{X}$ given by $h(z) = \mathbb{E}[Y|Z = z]$.

Example 1.2 (Example 1.1 continued). *Setting $p = 1/2$ we see that*

$$\begin{aligned}\mathbb{E}[(X_3)^2|X_2 = 2] &= \sum_{i \in \mathbb{N}} i\mathbb{P}[(X_3)^2 = i|X_2 = 2]\\ &= (1)^2\mathbb{P}[X_3 = 1|X_2 = 2] + (3)^2\mathbb{P}[X_3 = 3|X_2 = 2] = 5\end{aligned}$$

Of course, X_2 can also take the value -2 and 0. For these values of X_2 we have

$$\begin{aligned}\mathbb{E}[(X_3)^2|X_2 = -2] =&(-1)^2\mathbb{P}[X_3 = -1|X_2 = -2] + (-3)^2\mathbb{P}[X_3 = -3|X_2 = -2] = 5\\ \mathbb{E}[(X_3)^2|X_2 = 0] =&(-1)^2\mathbb{P}[X_3 = -1|X_2 = 0] + (1)^2\mathbb{P}[X_3 = 1|X_2 = 0] = 1\end{aligned}$$

Hence $\mathbb{E}[(X_3)^2|X_2] = h(X_2)$ where

$$h(x) = \begin{cases} 5 & \text{if } x = \pm 2\\ 1 & \text{if } x = 0\end{cases}$$

By clever rearrangement one does not always have to calculate the function $\mathbb{E}[Y|Z]$ so explicitly. Consider the following examples.

$$\begin{aligned}\mathbb{E}[X_7|X_6] = \mathbb{E}\left[\sum_{i=1}^{7}\omega_i|X_6\right] &= \mathbb{E}\left[\sum_{i=1}^{6}\omega_i + \omega_7|X_6\right]\\ =&\mathbb{E}\left[X_6 + \omega_7|X_6\right] = \mathbb{E}[X_6|\mathcal{F}_6] + \mathbb{E}[\omega_7|\mathcal{F}_6]\\ =&X_6 + \mathbb{E}[\omega_7] = X_6\end{aligned}$$

since $\mathbb{E}[\omega_7] = 0$. We can also do a similar calculation for the previous example.

$$\begin{aligned}\mathbb{E}[X_3^2|X_2] &= \mathbb{E}[(X_2 + \omega_3)^2|X_2] = \mathbb{E}[X_2^2 + 2\omega_3 X_2 + \omega_3^2|X_2]\\ &= \mathbb{E}[X_2^2] + 2\mathbb{E}[\omega_3]\mathbb{E}[X_2] + \mathbb{E}[\omega_3^2] = X_2^2 + 1\end{aligned}$$

since $\mathbb{E}[\omega_3] = 0$ and $\mathbb{E}[\omega_3^2] = 1$. Compare this to the definition of h given above.

x

2. Uncountable Probability Spaces

If we consider Example 1.1 in the case $N = \infty$ (or even worse if we imagine our stochastic process to live on the continuous interval $[0, 1]$) we need to consider Ω which have uncountably many points. To illustrate the difficulties one can encounter in this setting let us condsider the following example:

Example 2.1. *Consider $\Omega = [0,1]$ and let $\mathbb{P}$ be the uniform probability distribution on Ω, i.e., that dmeasure that associates the same probability to each on the points in Ω. We immediately see that in order to have $\mathbb{P}[\Omega]$ to be finite we must have $\mathbb{P}[\omega] = 0$ for all $\omega \in \Omega$, as otherwise*

$$\mathbb{P}[\Omega] = \mathbb{P}\left[\bigcup_{\omega\in\Omega} \omega\right] = \sum_{\omega\in\Omega} \mathbb{P}[\omega] = \infty\,.$$

For this reason it is not sufficient anymore to simply assign a probability to each point $\omega \in \Omega$ as we did before. We have to assign a probability to sets*:*

$$\mathbb{P}[(a,b)] = b - a \qquad \textit{for } 0 \leqslant a \leqslant b \leqslant 1\,.$$

To handle the setting such as the one introduced above completely rigorously we need ideas from basic measure theory. However if one is willing to except a few formal rules of manipulation, we can proceed with learning basic stochastic calculus without needing to distract ourselves with too much measure theory.

As we did in the previous section, we can define a real random variable as a function $X : \Omega \to \mathbb{R}$. To define the measure associated by $\mathbb{P}$ to the values of this random variable we specify its Cumulative Distribution Function (CDF) $F(x)$ defined as $\mathbb{P}[X \leqslant x] = F(x)$. We say that a $\mathbb{R}$-valued random variable X is a *continuous random variable* if there exists an (absolutely continuous) *density function* $\rho\colon \mathbb{R} \to \mathbb{R}$ so that

$$\mathbb{P}[X \in [a,b]] = \int_a^b \rho(x)dx$$

for any $[a,b] \subset \mathbb{R}$. By the fundamental theorem of calculus we see that $\rho(x)$ satisfies $\rho(x) = F'(x)$. More generally a $\mathbb{R}^n$-valued random variable X is called a *continuous random variable* if there exists a density function $\rho\colon \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ so that

$$\mathbb{P}[X \in [a,b]] = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \rho(x_1,\ldots,x_n)\,dx_1\cdots dx_n = \int_{[a,b]} \rho(x)dx = \int_{[a,b]} \rho(x)\mathrm{Leb}(dx)$$

for any $[a,b] = \prod [a_i, b_i] \subset \mathbb{R}^n$. The last two expressions are just different ways of writing the same thing. Here we have introduced the notation $\mathrm{Leb}(dx)$ for the standard Lebesgue measure on $\mathbb{R}^n$ given by $dx_1 \cdots dx_n$.

If X and Y are $\mathbb{R}^n$-valued and $\mathbb{R}^m$-valued random variables, respectively, then the vector (X,Y) is again a continuous $\mathbb{R}^{n\times m}$-valued random variable which has a density which is called the *joint probability density function* (joint density for short) of X and Y. If Y has density ρ_Y and ρ_{XY} is the joint density of X and Y we can define

$$\mathbb{P}[X \in A | Y = y] = \int_A \frac{\rho_{XY}(x,y)}{\rho_Y(y)} dx\,. \tag{2.5}$$

Hence X given $Y = y$ is a new continuous random variable with density $x \mapsto \frac{\rho_{XY}(x,y)}{\rho_Y(y)}$ for a fixed y.

Finally, analogously to the countable case we define the expectation of a continuous random variable with density ρ by

$$\mathbb{E}[h(X)] = \int_{\mathbb{R}^n} h(x)\rho(x)dx\,.$$

The conditional expectation is defined using the density (2.5).

Definition 2.2. *A real-valued random variable X is* Gaussian *with mean μ and variance σ^2 if*

$$\mathbb{P}[X \in A] = \int_A \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx\,.$$

If a random variable has this distribution we will write $X \sim \mathcal{N}(\mu, \sigma^2)$ *. More generally we say that a* $\mathbb{R}^n$*-valued random variable* X *is* Gaussian *with mean* $\mu \in \mathbb{R}^n$ *and SPD covariance matrix* $\Sigma \in GL(\mathbb{R}^n)$ *if*

$$\mathbb{P}[X \in A] = \int_A \frac{1}{\sqrt{2\pi \det(\Sigma)}} \exp\left[-\frac{(x-\mu)^\top \Sigma^{-1} (x-\mu)}{2}\right] dx\,.$$

While many calculations can be handled satisfactorily at this level, we will soon see that we need to consider random variables on much more complicated spaces such as the space of real-valued continuous functions on the time interval $[0,T]$ which will be denoted $C([0,T];\mathbb{R})$. To give all of the details in such a setting would require a level of technical detail which we do not wish to enter into on our first visit to the subject of stochastic calculus. If one is willing to "suspend a little disbelief" one can learn the formal rule of manipulation, much as one did when one first learned regular calculus. The technical details are important but better appreciated after one fist has the big picture.

3. General Probability Spaces and Sigma Algebras

To this end, we will introduce the idea of a sigma algebra (usually written σ-algebra or σ-field in [Klebaner]). In Section 1, we defined our probability measures by beginning with assigning a probability to each $\omega \in \Omega$. This was fine when Ω was finite or countably infinite. However, as we have seen in Example 2.1, when Ω is uncountable as in the case of picking a uniform point from the unit interval ($\Omega = [0,1]$), the probability of any given point must be zero. Otherwise the sum of all of the probabilities would be ∞ since there are infinitely many points and each of them has the same probability as no point is more or less likely than another.

This is only the tip of the iceberg. There are many more complicated issues. The solution is to fix a collection of subsets of Ω about which we are "allowed" to ask "what is the probability of this event?". We will be able to make this collection of subsets very large, but it will *not*, in general, contain *all* of the subsets of Ω in situations where Ω is uncountable. This collections of subsets is called the σ-algebra. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ of an outcome space Ω, a probability measure $\mathbb{P}$ and a σ-algebra $\mathcal{F}$ is called a *Probability Space*. For any event $A \in \mathcal{F}$, the "probability of this event happening" is well defined and equal to $\mathbb{P}[A]$. A subset of Ω which is not in $\mathcal{F}$ might not have a well defined probability. Essentially all of the event you will think of naturally will be in the σ-algebra with which we will work. In light of this, it is reasonable to ask why we bring them up at all. It turns out that σ-algebras are a useful way to "encode the information" contained in a collection of events or random variables. This idea and notation is used in many different contexts. If you want to be able to read the literature, it is useful to have a operational understanding of σ-algebras without entering into the technical detail.

Before attempting to convey any intuition or operational knowledge about σ-algebras we give the formal definitions since they are short (even if unenlightening).

Definition 3.1. *Given a set* Ω*, a* σ-algebra $\mathcal{F}$ *is a collection of subsets of* Ω *such that*

i) $\Omega \in \mathcal{F}$

ii) $A \in \mathcal{F} \implies A^c = \Omega \backslash A \in \mathcal{F}$

iii) *Given* $\{A_n\}$ *a countable collection of subsets of* $\mathcal{F}$*, we have* $\bigcup_{i=1}^\infty A_i \in \mathcal{F}$*.*

In this case the pair $(\Omega, \mathcal{F})$ *are referred to as a* measurable space

For us a σ*-algebra is the embodiment of information.*

Example 3.2. *If $\Omega = \mathbb{R}^n$ or any subset of it, we talk about the Borel σ-algebra as the σ-algebra generated by all of the intervals $[a, b]$ with $a, b \in \Omega$. This σ-algebra contains essentially any event you would think about in most all reasonable problems. Using (a, b), or $[a, b)$ or $(a, b]$ or some mixture of them makes no difference.*

Given any collection of subsets $\mathcal{G}$ of Ω we can talk about the "σ-algebra generated by $\mathcal{G}$" as simply what we get by taking all of the elements of $\mathcal{G}$ and exhaustively applying all of the operations listed above in the definition of a σ-algebra. More formally,

Definition 3.3. *Given Ω and F a collection of subsets of Ω, $\sigma(F)$ is the σ-algebra generated by F. This is defined as the smallest (in terms of numbers of sets) σ-algebra which contains F. Intuitively $\sigma(F)$ represents all of the probability data contained in F.*

Example 3.4 (Example 1.1 continued). *We define*

$$F_1 = \{\{\omega \in \Omega \ : \ \omega_1 = 1\}, \{\omega \in \Omega \ : \ \omega_1 = -1\}\},$$

as a division of the possible outcomes fixing ω_1. This collection of sets generates a σ-algebra on Ω, given by

$$\mathcal{F}_1 := \{\emptyset, \Omega, \{\omega \in \Omega \ : \ \omega_1 = 1\}, \{\omega \in \Omega \ : \ \omega_1 = -1\}\}, \tag{2.6}$$

representing the information we have on the process knowing ω_1.

To complete our measurable space $(\Omega, \mathcal{F})$ into a probability space we need to add a probability measure. Since we will not build our measure from its definition on individual $\omega \in \Omega$ as we did in Section 1 we will instead assume that it satisfies certain reasonable properties which follow from this construction in the countable or finite case. The fact that the following assumptions is all that is needed would be covered in a measure theoretical probability or analysis class.

Definition 3.5. *A measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a* probability measure *if*

i) $\mathbb{P}[\Omega] = 1$,
ii) $\mathbb{P}[A^c] = 1 - \mathbb{P}[A]$ *for all* $A \in \mathcal{F}$.
iii) *Given $\{A_i\}$ a finite collection of pairwise disjoint sets in $\mathcal{F}$, $\mathbb{P}\left[\bigcup_{i=1}^n A_i\right] = \sum_{i=1}^n \mathbb{P}[A_i]$,*

In this case the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is referred to as a probability space

Definition 3.6. *If $(\Omega, \mathcal{F})$ and $(\mathbf{X}, \mathcal{B})$ are measurable spaces, then $\xi : \Omega \to \mathbf{X}$ is a $\mathbf{X}$-valued* random variable *if for all $B \in \mathcal{B}$ we have $\xi^{-1}(B) \in \mathcal{F}$. In other words, admissible events in $\mathbf{X}$ get mapped to admissible events in Ω.*

Given any events A and B in $\mathcal{F}$, we define the conditional probability just as before, namely

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$

Given real-valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define the expected value of X in a way analogous to before:

$$\mathbb{E}(X) = \int_\Omega X(\omega)\mathbb{P}(d\omega).$$

We will take for granted that this integral makes sense. However, it follows from the general theory of measure spaces.

Definition 3.7. *Given a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a measurable space $(\mathbf{X}, \mathcal{B})$, we define the σ-algebra generated by the random variable X as*

$$\sigma(X) = \sigma(\{X^{-1}(B) | B \in \mathcal{B}\}).$$

The idea is that $\sigma(X)$ contains all of the information contained in X. If an event is in $\sigma(X)$ then whether this event happens or not is completely determined by knowing the value of the random variable X.

EXAMPLE 3.8 (Example 1.1 continued). *By definition (2.1), since $X_1 = \omega_1$ the σ-algebra generated by the random variable X_1 is $\sigma(X_1) = \mathcal{F}_1$ from (2.6). However, the σ-algebra generated by $X_2 = \omega_1 + \omega_2$ is given by*

$$\sigma(X_2) = \{\emptyset, \Omega, \{\omega \in \Omega \,:\, (\omega_1, \omega_2) = (1,1)\}, \{\omega \in \Omega \,:\, (\omega_1, \omega_2) = (-1,-1)\}, \{\omega \in \Omega \,:\, \omega_1 + \omega_2 = 0\}\}\,.$$

Note that this σ-algebra is different than $\mathcal{F}_2 = \sigma(\{\{w \in \Omega \,:\, (w_1, w_2) = (s_1, s_2)\} \,:\, s_1, s_2 \in \{-1,1\}\})$. Indeed, knowing the value of X_2 is not always sufficient to know the value of $\omega_1 = X_1$. Contrarily, knowing the value of (ω_1, ω_2) (contained in $\mathcal{F}_2$) definitely implies that you know the value of X_2. In other words, (the information of) $\sigma(X_2)$ is contained in $\mathcal{F}_2$, concisely $\sigma(X_2) \subset \mathcal{F}_2$.

Now compare $\sigma(X_2)$ and $\sigma(Y)$ where $Y = X_2^2$. Lets consider three events $A = \{X_2 = 2\}$, $B = \{X_2 = 0\}$, $C = \{X_2$ is even$\}$. Clearly all three events are in the σ-algebra generated by X_2 (i.e. $\sigma(X_2)$) since if you know that value of X_2 then you ***always*** *know whether the events happen or not. Next notice that $B \in \sigma(Y)$ since if you know that $Y = 0$ then $X_2 = 0$ and if $Y \neq 0$ then $X_2 \neq 0$. Hence no mater what the value of Y is knowing it you can decide if $X_2 = 0$ or not. However, knowing the value of Y does not* ***always*** *tell you if $X_2 = 2$. It does sometimes, but not always. If $Y = 0$ then you know that $X_2 \neq 0$. However if $Y = 4$ then X_2 could be equal to either 2 or -2. We conclude that $A \notin \sigma(Y)$ but $B \in \sigma(Y)$. Since X_2 is always even, we do not need to know any information to decide C and it is in fact in both $\sigma(X_2)$ and $\sigma(Y)$. In fact, $C = \Omega$ and Ω is in any σ-algebra since by definition Ω and the empty set $\emptyset$ are always included. Lastly, since whenever we know X_2 we know Y, it is clear that $\sigma(X_2)$ contains all of the information contained in $\sigma(Y)$. In fact it follows from the definition and the fact that $\sigma(Y) \subset \sigma(X_2)$. To say that one σ-algebra is contained in another is to say that the second contains all of the information of the first and possibly more.*

DEFINITION 3.9. *We say that a real-valued random variable X is* measurable *with respect to σ-algebra $\mathcal{G}$ if every set of the form $X^{-1}([a,b])$ is in $\mathcal{G}$.*[1]

Speaking intuitively, a random variable is measurable with respect to a given σ-algebra if the information in the σ-algebra is **always** sufficient to fix the value of the random variable. Of course the random variable X is always measurable with respect to $\sigma(X)$. More specifically, $\sigma(X)$ is the smallest σ-algebra $\mathcal{G}$ on Ω such that X is $\mathcal{G}$-measurable. In the previous example, Y is measurable with respect to $\sigma(X_2)$ since knowing the value of X_2 fixes the value of Y.

DEFINITION 3.10. *If a random variable X is measurable with respect to a σ-algebra $\mathcal{F}$ then we will write $X \in \mathcal{F}$. While this is a slight abuse of notation, it will be very convenient.*

EXAMPLE 3.11. *Let X be a random variable taking values in $[-1,1]$. Let g be the function from $[-1,1] \to \{-1,1\}$ such that $g(x) = -1$ if $x \leqslant 0$ and $g(x) = 1$ if $x > 0$. Define the random variable Y by $Y(\omega) = g(X(\omega))$. Hence Y is a random variable talking values in $\{-1,1\}$. Let $\mathcal{F}_Y$ be the σ-algebra generated by the random variable Y. That is $\mathcal{F}_Y = \sigma(Y) := \{Y^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}$. In this case, we can figure out exactly what $\mathcal{F}_Y$ looks like. Since Y takes on only two values, we see that for any subset B in $\mathcal{B}(\mathbb{R})$(the Borel σ-algebra of $\mathbb{R}$)*

[1] If X is a random variable taking values in a measurable space $(\mathbf{X}, \mathcal{B})$ (recall that $\mathcal{B}$ is a σ-algebra over $\mathbf{X}$) then we require that $X^{-1}(B) \in \mathcal{G}$ for all $B \in \mathcal{B}$.

$$Y^{-1}(B) = \begin{cases} Y^{-1}(-1) := \{\omega : Y(\omega) = -1\} & \text{if } -1 \in B, 1 \notin B \\ Y^{-1}(1) := \{\omega : Y(\omega) = 1\} & \text{if } 1 \in B, -1 \notin B \\ \varnothing & \text{if } -1 \notin B, 1 \notin B \\ \Omega & \text{if } -1 \in B, 1 \in B \end{cases}$$

Thus $\mathcal{F}_Y$ consists of exactly four sets, namely $\{\varnothing, \Omega, Y^{-1}(-1), Y^{-1}(1)\}$. For a function $f : \Omega \to \mathbb{R}$ to be measurable with respect the σ-algebra $\mathcal{F}_Y$, the inverse image of any set $B \in \mathcal{B}(\mathbb{R})$ must be one of the four sets in $\mathcal{F}_Y$. This is another way of saying that f must be constant on both $Y^{-1}(-1)$ and $Y^{-1}(1)$. Note that together $Y^{-1}(-1) \cup Y^{-1}(1) = \Omega$.

We now re-examine the idea of a conditional expectation of a random variable with respect to a σ-algebra. To do so, we introduce the indicator function.

Definition 3.12. *Given $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $A \in \mathcal{F}$, the indicator function of A is*

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & \textit{otherwise} \end{cases} \tag{2.7}$$

and is a measurable function. Fixing a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we have

Proposition 3.13. *If X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[|X|] < \infty$, and $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, then there is a unique random variable Y on $(\Omega, \mathcal{G}, \mathbb{P})$ such that*

i) $\mathbb{E}[|Y|] < \infty$,

ii) $\mathbb{E}[\mathbb{1}_A Y] = \mathbb{E}[\mathbb{1}_A X]$ *for all* $A \in \mathcal{G}$.

Definition 3.14. *We define the* conditional expectation *with respect to a σ-algebra $\mathcal{G}$ as the unique random variable Y from Proposition 3.13, i.e., $\mathbb{E}[X|\mathcal{G}] := Y$.*

The intuition behind Proposition 3.14 is that the conditional expectation wrt a σ-algebra $\mathcal{G} \subset \mathcal{F}$ of a random variable $X \in \mathcal{F}$ is that random variable $Y \in \mathcal{G}$ that is equivalent or identical (in terms of expected value, or predictive power) to X given the information contained in $\mathcal{G}$. In other words, $Y = \mathbb{E}[X|\mathcal{G}]$ is **the best approximation** of the value of X given the information in $\mathcal{G}$. The previous definition of conditional expectation wrt a fixed set of events is obtained by *evaluating* the random variable $\mathbb{E}[X|\mathcal{G}]$ on the events of interest, *i.e.*, by *fixing* the events in $\mathcal{G}$ that may have occurred.

When we condition on a random variable we are really conditioning on the *information* that random variable is giving to us. In other words, we are conditioning on the σ-algebra generated by that random variable:

$$\mathbb{E}\left[X|Z\right] := \mathbb{E}\left[X|\sigma(Z)\right].$$

As in the discrete case, one can show that there exists a function h : Range$(Z) \to \mathbf{X}$ such that

$$\mathbb{E}[Y|Z(\omega)] := h(Z(\omega)),$$

and hence we can think about the conditional expectation as a function of $Z(\omega)$. In particular, this allows to define

$$\mathbb{E}[Y|Z = z] := h(z).$$

EXAMPLE 3.15. *When the set Ω is countable, we can write every random variable Y on Ω as*

$$Y(\omega) = \sum_{y \in Range(Y)} y \mathbb{1}_{Y=y}(\omega) .$$

Consequently, writing

$$\mathbb{E}[X|Z](\omega) = \sum_{z \in Range(Z)} \mathbb{E}[X|Z = z] \mathbb{1}_{Z=z}(\omega) ,$$

we obtain

$$\mathbb{E}[\mathbb{1}_{Z=z}(\omega) X(\omega)] = \mathbb{E}\left[\mathbb{1}_{Z=z}(\omega) \mathbb{E}[X|Z = z]\right] = \mathbb{E}[X|Z = z] \cdot \mathbb{E}[\mathbb{1}_{Z=z}(\omega)]$$

Now, recognizing that $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}[A]$, if $\mathbb{P}[Z = z] \neq 0$ we finally obtain

$$\mathbb{E}[X|Z = z] = \frac{\mathbb{E}[\mathbb{1}_{Z=z}(\omega) X(\omega)]}{\mathbb{P}[Z = z]} = \sum_{x \in Range(X)} \frac{x \mathbb{P}[Z = z, X = x]}{\mathbb{P}[Z = z]} ,$$

and recover (2.3).

We now list some properties of the conditional expectation:

- **Linearity:** for all $\alpha, \beta \in \mathbb{R}$ we have

$$\mathbb{E}[\alpha X + \beta Y|\mathcal{G}] = \alpha \mathbb{E}[X|\mathcal{G}] + \beta \mathbb{E}[Y|\mathcal{G}] ,$$

- if X is $\mathcal{G}$-measurable then

$$\mathbb{E}[XY|\mathcal{G}] = X \mathbb{E}[Y|\mathcal{G}] .$$

  Intuitively, since $X \in \mathcal{G}$ (X is measurable WRT the σ-algebra $\mathcal{G}$), the best approximation of X on the sets contained in $\mathcal{G}$ is X itself, so we do not need to approximate it.
- **Tower property:** if $\mathcal{G}$ and $\mathcal{H}$ are both σ-algebras with $\mathcal{G} \subset \mathcal{H}$, then

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{H}\right]|\mathcal{G}\right] = \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]|\mathcal{H}\right] = \mathbb{E}\left[X|\mathcal{G}\right] .$$

  Since $\mathcal{G}$ is a smaller σ-algebra, the functions which are measurable with respect to it are contained in the space of functions measurable with respect to $\mathcal{H}$. More intuitively, being measure with respect to $\mathcal{G}$ means that only the information contained in $\mathcal{G}$ is left free to vary. $\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{H}\right]|\mathcal{G}\right]$ means first give me your best guess given only the information contained in $\mathcal{H}$ as input and then reevaluate this guess making use of only the information in $\mathcal{G}$ which is a subset of the information in $\mathcal{H}$. Limiting oneself to the information in $\mathcal{G}$ is the bottleneck so in the end it is the only effect one sees. In other words, once one takes the conditional expectation with respect to a smaller σ algebra one is loosing information. Therefore, by doing $\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]|\mathcal{H}\right]$ one is loosing information (in the innermost expectation) that cannot be recovered by the second one.
- **Jensen's inequality** If $g : I \to \mathbb{R}$ is convex[2] on $I \subseteq \mathbb{R}$ for a random variable $X \in \mathcal{G}$ with range(X) $\subseteq I$ we have

$$g(\mathbb{E}[X]|\mathcal{G}) \leqslant \mathbb{E}[g(X)|\mathcal{G}] ,$$

- **Chebysheff inequality** For a random variable $X \in \mathcal{G}$ we have that for any $\lambda > 0$

$$\mathbb{P}[|X| > \lambda|\mathcal{G}] \leqslant \frac{\mathbb{E}[|X|\,|G]}{\lambda} ,$$

[2]A function g is convex on $I \subseteq \mathbb{R}$ if for all $x, y \in I$ with $[x, y] \subseteq I$ and for all $\lambda \in [0, 1]$ one has $g(\lambda x + (1 - \lambda)y) \leqslant \lambda g(x) + (1 - \lambda)g(y)$

- **Optimal approximation** The conditional expectation with respect to a σ-algebra $\mathcal{G} \subset \mathcal{F}$ by

$$\mathbb{E}[Y|\mathcal{G}] = \underset{Z \text{ meas w.r.t. } \mathcal{G}}{\operatorname{argmin}} \mathbb{E}[Y - Z]^2 \tag{2.8}$$

This should be thought of as the best guess of the value of Y given the information in $\mathcal{G}$.

Example 3.16 (Example 3.11 continued). *In the previous example, $\mathbb{E}\{X|\mathcal{F}_Y\}$ is the best approximation to X which is measurable with respect to $\mathcal{F}_Y$, that is constant on $Y^{-1}(-1)$ and $Y^{-1}(1)$.In other words, $\mathbb{E}\{X|\mathcal{F}_Y\}$ is the random variable built from a function h_{min} composed with the random variable Y such that the expression*

$$\mathbb{E}\left\{(X - h_{min}(Y))^2\right\}$$

is minimized. Since $Y(\omega)$ takes only two values in our example, the only details of h_{min} which mater are its values at 1 and -1. Furthermore, since $h_{min}(Y)$ only depends on the information in Y, it is measurable with respect to $\mathcal{F}_Y$. If by chance X is measurable with respect to $\mathcal{F}_Y$, then the best approximation to X is X itself. So in that case $\mathbb{E}\{X|\mathcal{F}_Y\}(\omega) = X(\omega)$.

In light of (2.8), we see that

$$\mathbb{E}[X|Y_1, \ldots, Y_k] = \mathbb{E}[X|\sigma(Y_1, \ldots, Y_k)]$$

This fits with our intuitive idea that $\sigma(Y_1, \ldots, Y_k)$ embodies the information contained in the random variables $Y_1, Y_2, \ldots Y_k$ and that $\mathbb{E}[X|\sigma(Y_1, \ldots, Y_k)]$ is our best guess at X if we only know the information in $\sigma(Y_1, \ldots, Y_k)$.

Definition 3.17. *Given $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $A, B \in \mathcal{F}$, we say that A and B are* independent *if*

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B] \tag{2.9}$$

Similarly, random variables $\{X_i\}$ are jointly independent *if for all C_i,*

$$\mathbb{P}[X_1 \in C_1 \text{ and } \ldots \text{ and } X_n \in C_n] = \prod_{i=1}^{n} \mathbb{P}[X_i \in C_i] \tag{2.10}$$

It is important to note that given two independent random variables X and Y one has

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y] \tag{2.11}$$

4. Distributions and Convergence of Random Variables

Definition 4.1. *We say that two $\mathbf{X}$-valued random variables X and Y* have the same distribution *or* have the same law *if for all bounded (measureable) functions $f : \mathbf{X} \to \mathbb{R}$ we have $\mathbb{E}[f(X)] = \mathbb{E}[f(Y)]$. This equivalence is sometimes written*

$$\mathrm{Law}(X) = \mathrm{Law}(Y) \tag{2.12}$$

Remark 4.2. *Either of the following are equivalent to two random variable X and Y on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ having the same distribution*.*

i) $\mathbb{E}[f(X)] = \mathbb{E}[f(Y)]$ *for all continuous f with compact support.*
ii) $\mathbb{P}[X \in A] = \mathbb{P}[Y \in A]$ *for all $A \in \mathcal{F}$.*

There are many ways a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ can converge to another random variable X:

Definition 4.3. *Let* $\{X_n\}_{n\in\mathbb{N}}$ *be a sequence of random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, and let* X *be a random variable on the same space. Then*

- **almost sure convergence** $\{X_n\}$ *converges to* X *almost surely if*
$$\mathbb{P}[\{w \in \Omega \ : \ \lim_{n\to\infty} X_n(\omega) = X(\omega)\}] = 1\,,$$
- **convergence in probability** $\{X_n\}$ *converges to* X *in probability if, for all* $\varepsilon > 0$
$$\lim_{n\to\infty} \mathbb{P}[\{w \in \Omega \ : \ |X_n(\omega) - X(\omega)| > \varepsilon\}] = 0\,,$$
- **weak convergence** $\{X_n\}$ *converges weakly (or in distribution) to* X *if, for all* $A \in \mathcal{B}$
$$\lim_{n\to\infty} \mathbb{P}[X_n(\omega) \in A] = \mathbb{P}[X(\omega) \in A]\,.$$

Remark 4.4. *The above definitions can be ordered by strength: we have the following implications*

almost sure convergence $\Rightarrow$ *convergence in probability* $\Rightarrow$ *weak convergence*.

Moreover, we note that in order to have convergence in distributions the two random variables do not need to live on the same probability space.

A useful method of showing that the distribution of a sequence of random variables converges to another is to consider the associated sequence of Fourier transforms, or the characteristic function of a random variable as it is called in probability theory.

Definition 4.5. *The characteristic function (or Fourier Transform) of a random variable* X *is defined as*
$$\psi(t) = \mathbb{E}[\exp(itX)]$$
for all $t \in \mathbb{R}$.

It is a basic fact that the characteristic function of a random variable uniquely determines its distribution. Furthermore, the following convergence theorem is a classical theorem from probability theory.

Theorem 4.6. *Let* X_n *be a sequence of real-valued random variables and let* ψ_n *be the associated characteristic functions. Assume that there exists a function* ψ *so that for each* $t \in \mathbb{R}$
$$\lim_{n\to\infty} \psi_n(t) = \psi(t)\,.$$
If ψ *is continuous at zero then there exists a random variable* X *so that the distribution of* X_n *converges to the distribution of* X*. Furthermore the characteristic function of* X *is* ψ.

Example 4.7. *Note that using a Fourier transform,*
$$\mathbb{E}[e^{i\lambda x}] = e^{i\lambda m - \frac{\sigma^2\lambda^2}{2}}\,,$$
for all λ*. Using this, we say that* $X = (X_1, \dots, X_k)$ *is a* k-dimensional Gaussian *if there exists* $m \in \mathbb{R}^k$ *and* R *a positive definite symmetric* $k \times k$ *matrix so that for all* $\lambda \in \mathbb{R}^k$ *we have*
$$\mathbb{E}[e^{i\lambda\cdot x}] = e^{i\lambda\cdot m - \frac{(R\lambda)\cdot\lambda}{2}}\,.$$

CHAPTER 3

Brownian Motion and Stochastic Processes

1. An Illustrative Example: A Collection of Random Walks

Fixing an $n \geqslant 0$, let $\{\xi_k^{(n)} : k = 1, \cdots, 2^n\}$ be a collection of independent random variables each distributed as normal with mean zero and variance 2^{-n}. For $t = k2^{-n}$ for some $k \in \{1, \cdots, 2^n\}$, we define

$$B^{(n)}(t) = \sum_{j=1}^{k} \xi_j^{(n)} . \tag{3.1}$$

for intermediate times $\in [0,1]$ not of the form $k2^{-n}$ for some k we define the function as the linear function connecting the two nearest points of the form $k2^{-n}$. In other words, if $t \in [s,r]$ were $s = k2^{-n}$ and $r = (k+1)2^{-n}$ then

$$B^{(n)}(t) = \frac{t-s}{2^n} B^{(n)}(s) + \frac{r-t}{2^n} B^{(n)}(r)$$

We will see momentarily that $B^{(n)}$ has the following properties independent of n:

i) $B^{(n)}(0) = 0$.
ii) $\mathbb{E}B^{(n)}(t) = 0$ for all $t \in [0,1]$.
iii) $\mathbb{E}|B^{(n)}(t) - B^{(n)}(s)|^2 = t - s$ for $0 \leqslant s < t \leqslant 1$ of the form $k2^{-n}$.
iv) The distribution of $B^{(n)}(t) - B^{(n)}(s)$ is Gaussian for $0 \leqslant s < t \leqslant 1$ of the form $k2^{-n}$.
v) The collection of random variable
$$B^{(n)}(t_i) - B^{(n)}(t_{i-1})$$
are mutually independent as long as $0 \leqslant t_0 < t_1 < \cdots < t_m \leqslant 1$ for some m and the t_i are of the form $k2^{-n}$.

The first property is clear since the sum in (3.1) is empty. The second property for $t = k2^{-n}$ follows from

$$\mathbb{E}B^{(n)}(t) = \sum_{j=1}^{k} \mathbb{E}\xi_j^{(n)} = 0$$

since $\mathbb{E}\xi_j^{(n)} =$. For general t, we have $\mathbb{E}B^{(n)}(t) = \frac{t-s}{2^n}\mathbb{E}B^{(n)}(s) + \frac{r-t}{2^n}\mathbb{E}B^{(n)}(r)$ with s, r of the form $k2^{-n}$ for different k.

To see the second moment calculation take $s = m2^{-m}$ and $t = k2^{n}$ and observe that

$$\begin{aligned}\mathbb{E}\left[|B^{(n)}(t) - B^{(n)}(s)|^2\right] &= \mathbb{E}\left[\Big(\sum_{j=m}^{k} \xi_j\Big)\Big(\sum_{\ell=m}^{k} \xi_\ell\Big)\right] = \sum_{j=m}^{k}\sum_{\ell=m}^{k} \mathbb{E}\left[\xi_j\xi_\ell\right] \\ &= \sum_{j=m}^{k} \mathbb{E}\left[\xi_j^2\right] + \sum_{j=m}^{k}\sum_{\substack{\ell=m\\ \ell\neq j}}^{k} \mathbb{E}\left[\xi_j\right]\mathbb{E}\left[\xi_\ell\right] = \sum_{j=m}^{k} 2^{-n} = k2^{-n} - m2^{-n} = t - s\end{aligned}$$

since ξ_j and ξ_ℓ are independent if $j \neq \ell$, and $\mathbb{E}\left[\xi_j\right] = 0$, and $\mathbb{E}\left[\xi_j^2\right] = 2^{-n}$.

Since $B^n(t)$ is just the sum of independent Gaussians, it is also distributed Gaussian with a mean and a variance which is just the sum of the individual means and variances respectively. Because for disjoint time intervals the differences of the $B^{(n)}(t_i) - B^{(n)}(t_{i-1})$ are sums over disjoint collections of $\xi_i's$, they are mutually independent.

Since all of these properties are independent of n it is tempting to think about the limit as $n \to \infty$ and the mesh becoming increasingly fine. It is not clear that such a limit would exist as the curves $B^{(n)}$ become increasingly "wiggly." We will see in fact that it does exist. We begin by taking an abstract perspective in the next sections though we will return to a more concrete perspective at the end.

2. General Stochastic Processes

Motivated by the example of the previous section, we pause to discuss the idea of a stochastic process more generally.

Definition 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(\mathbf{X}, \mathcal{B})$ be a measurable space. Also let $\mathcal{T}$ be an indexing set which for our purposes will typically be $\mathbb{R}, \mathbb{R}_+, \mathbb{N}$, or $\mathbb{Z}$. Suppose that for each $t \in \mathcal{T}$ we have $X_t : \Omega \to \mathbf{X}$ a measurable function. Then the set $\{X_t\}$ is a* stochastic process *on T with values in $\mathbf{X}$. Also, given $\omega \in \Omega$, $\xi_t(\omega) : \mathcal{T} \to \mathbf{X}$ is called a* path *or* trajectory *of ξ_t.*

Remark 2.2. *Commonly used notations for stochastic processes include $\{X_t\}, X_t(\omega), X_t(\omega), \dots$*

We want to define a type of equivalence between stochastic processes. But first of recall the following notion of equivalence of random variables.

Definition 2.3. *We say that two stochastic processes* have the same distribution *or* have the same law *if for all $t_1 < \cdots < t_n \in \mathcal{T}$ we have*

$$\mathrm{Law}(X_{t_1}, \dots, X_{t_n}) = \mathrm{Law}(Y_{t_1}, \dots, Y_{t_n})$$

where we think of the vector $(X_{t_1}, \dots X_{t_n})$ as a random variable taking values in the product space $\mathbf{X}^n$.

We would like to state a nice extension theorem for constructing stochastic processes, but first we need a definition.

Definition 2.4. *Given a set of finite dimensional distributions $\{\mu\}$ over an indexing set $\mathcal{T}$ on $\mathbf{X}$ we say that the set is* compatible *if*

i) *For all $t_1 < \cdots < t_{m+1} \in \mathcal{T}$ and $A_1, \dots, A_m \in \mathcal{B}$ we have*

$$\mu_{t_1 \dots t_m}(A_1, \dots, A_m) = \mu_{t_1 \dots t_{m+1}}(A_1, \dots, A_m, \mathbf{X})$$

ii) *For all $t_1 < \cdots < t_m$, $A_1, \dots, A_m \in \mathcal{B}$, and σ a permutation on m letters, we have*

$$\mu_{t_1 \dots t_m}(A_1, \dots, A_m) = \mu_{t_{\sigma(1)} \dots t_{\sigma(m)}}\left(A_{\sigma(1)}, \dots, A_{\sigma(m)}\right)$$

The first condition is roughly saying that if one considers a null condition (the total space) in a higher-dimensional measure, one gets the same result without the null condition in the lower dimensional measure. The second condition is saying that the order of the indexing of the μ doesn't matter.

Remark 2.5. *The first of the above two requirements is sometimes called the* Chapman-Kolmogorov *equation.*

Theorem 2.6 (Kolmogorov Extension Theorem). *Given a set of compatible finite dimensional distributions* $\{\mu_{t_1 \dots t_m}\}$ *with indexing set* $\mathcal{T}$, *there exists a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a stochastic process* $\{X_t\}$ *so that* X_t *has the required finite dimensional distributions, i.e. for all* $t_1 < \cdots < t_m \in \mathcal{T}$ *and* $A_1, \dots A_m \in \mathcal{B}$ *we have*

$$\mathbb{P}[X_{t_1} \in A_1 \text{ and } \dots \text{ and } X_{t_m} \in A_m] = \mu_{t_1 \dots t_m}(A_1, \dots A_m) \tag{3.2}$$

3. Definition of Brownian motion (Wiener Process)

Looking back at Section 1, the list of properties of $B^{(n)}$ suggest a reasonable collection of compatible finite distributions. Namely independent increments with each increment distributed normally with mean zero and variance proportional to the time interval. and we make the following definition.

Definition 3.1. *Standard Brownian motion* $\{B_t\}$ *is a stochastic process on* $\mathbb{R}$ *such that*

i) $B_0 = 0$ *almost surely (i.e.* $\mathbb{P}\left[\{\omega \in \Omega \ : \ B_0 \neq 0\}\right] = 0$*),*

ii) B_t *has independent increments: for any* $t_1 < t_2 < \ldots < t_n$,
$B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ *are independent,*

iii) *The increments* $B_t - B_s$ *are Gaussian random variables with mean* 0 *and variance given by the length of the interval:*

$$Var(B_t - B_s) = |t - s| \,.$$

Since this is a compatible collection of finite dimensional distributions, Theorem 2.6 guarantees the existence of the process we have described.

Looking back at $B^{(n)}$ from (3.1) it might be reasonable to hope that the Brownian motion $\{B_t\}$ defined above might be a continuous function of time. Notice that the above definition makes no mention of the continuity. The following definition makes it clear what we mean by continuous.

Definition 3.2. *A stochastic process is* continuous *if all the trajectories* $t \to X_t(\omega)$ *are continuous.*

It turns out that the finite dimensional distributions can not *guarantee* that a stochastic process is almost surely continuous. However they can imply that it is possible for a given process to be continuous.

Definition 3.3. *A stochastic process* $\{X_t\}$ *is a version (or modification) of a second stochastic process* $\{Y_t\}$ *if for all* t, $\mathbb{P}[X_t = Y_t] = 1$. *Notice that this is a symmetric relation.*

Theorem 3.4 (Kolmogorov Continuity Theorem (a version)). *Suppose that a stochastic process* $\{X_t\}$, $t \geqslant 0$ *satisfies the estimate:*
for all $T > 0$ *there exist positive constants* α, β, D *so that*

$$\mathbb{E}[|X_t - X_s|^\alpha] \leqslant D|t-s|^{1+\beta} \quad \forall t, s \in [0, T], \tag{3.3}$$

then there exist a version of X_t *which is continuous.*

Remark 3.5. *The estimate in* (3.3) *holds for a Brownian motion. We give the details in one-dimension. First recall that if* X *is a Gaussian random variable with mean* 0 *and variance* σ^2 *then* $\mathbb{E}[X^4] = 3\sigma^4$. *Applying this to Brownian motion we have* $\mathbb{E}|B_t - B_s|^4 = 3|t-s|^2$ *and conclude that* (3.3) *holds with* $\alpha = 4$, $\beta = 1$, $D = 3$. *Hence it is not incompatible with the all ready assumed properties of Brownian motion to assume that* B_t *is continuous almost surely.*

Remark 3.5 shows that continuity is a fundamental attribute of Brownian motion. In fact we have the following second (and equivalent) definition of Brownian motion which assumes a form of continuity as a basic assumption, replacing other assumptions.

THEOREM 3.6. *Let B_t be a stochastic process such that the following conditions hold:*

i) $\mathbb{E}(B_1^2 = constant$,
ii) $B_0 = 0$ *almost surely,*
iii) $B_{t+h} - B_t$ *is independent of* $\{B_s : s \leqslant t\}$.
iv) *The distribution of* $B_{t+h} - B_t$ *is independent of* $t \geqslant 0$ *(stationary increments),*
v) *(Continuity in probability.) For all* $\delta > 0$,

$$\lim_{h \to 0} \mathbb{P}[|B_{t+h} - B_t| > \delta] = 0$$

then B_t is Brownian motion. When $\mathbb{E}B(1)^2 = 1$ we call it standard Brownian motion.

The process introduced above can be straightforwardly generalized to n dimensions:

DEFINITION 3.7. *n-dimensional Standard Brownian motion $\{B_t\}$ is a stochastic process on $\mathbb{R}^n$ such that*

i) $B_0 = 0$ *almost surely (i.e.* $\mathbb{P}[\{\omega : B_0 \neq 0\}] = 0$*),*
ii) B_t *has independent increments: for any* $t_1 < t_2 < \ldots < t_n$,
$B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent,*
iii) *The increments* $B_t - B_s$ *are Gaussian random variables with mean* 0 *and variance given by the length of the interval: denoting by* $(B_t)_i$ *the i-th component of* B_t

$$Var((B_t)_i - (B_s)_j) = \begin{cases} |t-s| & \text{if } i = j \\ 0 & \text{else} \end{cases} .$$

4. Constructive Approach to Brownian motion

Returning to the construction of Section 1, one might be tempted to hope that the random walks $B^{(n)}$ converge to a Brownian motion B_t as $n \to \infty$. While this is true that the distribution of $B^{(n)}$ converges weakly to that of B_t as $n \to \infty$, a moments reflection shows that there is not hope that the sequence converges almost surely since $B^{(n)}$ and $B^{(n+1)}$ have no relation for a given realization of the underlying random variable $\{\xi_k^{(n)}\}$.

We will now show that by cleverly rearranging the randomness we can construct a new sequence of random walks $W^{(n)}(t)$ so that the stochastic process $W^{(n)}$ has the same distribution as $B^{(n)}$ yet $W^{(n)}$ will converge almost surely to a realization of Brownian motion.

We begin by defining a new collection of random variables $\{\eta_k^{(n)}\}$ from the $\{\xi_k^{(n)}\}$. We define $\eta_1^{(0)}$ to be a normal random variable with mean 0 and variance 1 which is independent of all of the ξ's. Then for $n \geqslant 0$ and $k \in \{1, \ldots, 2^k\}$ we define

$$\eta_{2k}^{(n+1)} = \frac{1}{2}\eta_k^{(n)} + \frac{1}{2}\xi_k^{(n)} \qquad \text{and} \qquad \eta_{2k-1}^{(n+1)} = \frac{1}{2}\eta_k^{(n)} - \frac{1}{2}\xi_k^{(n)}$$

Since each $\eta_k^{(n)}$ is the sum of independent Gaussian random variables they are themselves Gaussian random variables. It is easy to see that $\eta_k^{(n)}$ is mean zero and has variance 2^{-n}. Since for any $n \geqslant 0$ and $j, k \in \{1, \ldots, 2^k\}$ with $j \neq k$, we see that $\mathbb{E}\eta_k^{(n)}\eta_j^{(n)} = 0$ and we conclude that because the variables are Gaussian that the collection of random variables $\{\eta_k^{(n)} : k \in \{1, \ldots, 2^k\}\}$ are mutually independent. Hence if we define

$$W^{(n)}(k2^{-n}) = \sum_{j=1}^{k} \eta_j^{(n)}$$

and at intermediate times as the value of the line connecting the two nearest points, then $W^{(n)}$ has the same distribution as $B^{(n)}$ from Section 1.

THEOREM 4.1. *With probability one, the sequence of functions* $\big(W^{(n)}(t)\big)(\omega)$ *on converges uniformly to a continuous function* $B_t(\omega)$ *as* $n \to \infty$*, and the process* $B_t(\omega)$ *is a Brownian motion on* $[0, 1]$.

PROOF. Now for $n \geqslant 0$ and $k \in \{1, \ldots, 2^k\}$ define

$$Z_k^{(n)} = \sup_{t \in [(k-1)2^{-n}, k2^{-n}]} \big|W^{(n)}(t) - W^{(n+1)}(t)\big|$$

and observe that

$$Z_k^{(n)} = \Big|\frac{1}{2}\eta_k^{(n)} - \eta_{2k-1}^{(n+1)}\Big| = \Big|\frac{1}{2}\xi_k^{(n)}\Big|$$

Since $\xi_k^{(n)}$ is normal with mean zero and variance $2^{-(n+2)}$, we have by Markov inequality that

$$\mathbb{P}[Z_k^{(n)} > \delta] \leqslant \frac{\mathbb{E}[|Z_k^{(n)}|^4]}{\delta^4} = \frac{3 \cdot 2^{-2(n+2)}}{\delta^4} .$$

In turn since the $\{Z_k^{(n)} : k = 1, \ldots, 2^n\}$ are mutually independent this implies that

$$\begin{aligned}\mathbb{P}\left[\sup_{t \in [0,1]} \big|W^{(n)}(t) - W^{(n+1)}(t)\big| > \delta\right] &= \mathbb{P}[\sup_k Z_k^{(n)} > \delta] = 2^n \mathbb{P}[Z_1^{(n)} > \delta] \\ &\leqslant 2^n \frac{3 \cdot 2^{-2(n+2)}}{\delta^4} := \psi(n, \delta) .\end{aligned}$$

Since $\psi(n, 2^{-n/5}) \sim c2^{-n/5}$ for some $c > 0$, we have that

$$\sum_{n=1}^{\infty} \mathbb{P}\Big[\sup_{t \in [0,1]} \big|W^{(n)}(t) - W^{(n+1)}(t)\big| > 2^{-n/5}\Big] < \infty .$$

Hence the Borel-Cantelli lemma implies that with probability one there exists a random $k(\omega)$ so that if $n \geqslant k$ then

$$\sup_{t \in [0,1]} \big|W^{(n)}(t) - W^{(n+1)}(t)\big| \leqslant 2^{-n/5} .$$

In other words with probability one the $\{W^{(n)}\}$ form a Cauchy sequence. Let B_t denote the limit. It is not hard to see that B_t has the properties that define Brownian motion. Furthermore since each $W^{(n)}$ is uniformly continuous and converge in the supremum norm to B_t, we conclude that with probability one B_t is also uniformly continuous. $\square$

5. Brownian motion has Rough Trajectories

In Section 4, we saw that Brownian motion could be seen as the limit of a ever roughening path. This leads us to wonder "how rough is Brownian motion?" We know it is continuous, but is it differentiable?

DEFINITION 5.1. *The (standard) p-th variation on the interval (s,t) of any continuous function f is defined to be*

$$V_p[f](s,t) = \sup_\Gamma \sum_k |f(t_{k+1}) - f(t_k)|^p \tag{3.4}$$

where the supremum is over all partitions

$$\Gamma = \{\{t_k\} \ : \ s = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}\,, \tag{3.5}$$

for some n.

For a given partition $\{t_k\}$ let us define the mesh wi dth of the partition to be

$$|\Gamma| = \sup_{0<k\leqslant N} |t_k - t_{k-1}|\,.$$

The variations of Brownian motion are finite only for a certain range of q:

PROPOSITION 5.2. *If B_t is a Brownian motion on the interval $[0,T]$ with $T<\infty$, then*

$$V_p[B](0,T) < \infty \qquad \text{a.s.} \qquad \text{if and only if} \qquad p > 2\,. \tag{3.6}$$

PROOF. See [**20, 13**] for details. □

The fact that large values of p imply boundedness of the quadratic variation may be surprising at first. However, this results from the fact that in order for the supremum in (3.4) to diverge for a continuous function we must consider a sequence Γ^N of partitions with diverging number of intervals. As these intervals become smaller, the variation that each of them captures becomes smaller. These small contribution become even smaller if they are raised to a power $p>1$, whence the (possible) convergence. This concept is in close relation with the one of *Hölder continuity*.

Notice that if a function f has a nice bounded derivative on the interval $[0,t]$ then $V_1[f](0,t)<\infty$ since

$$|f(t_k) - f(t_{k+1})| = \left|{t_k}^{t_{k+1}} f'(s)\,\mathrm{d}s\right| \leqslant \big(\sup_{s\in[0,t]} |f'(s)|\big)\,|t_{k+1}-t_k|\,,$$

we see that

$$V_1[f](0,t) \leqslant \big(\sup_{s\in[0,t]} |f'(s)|\big) t\,.$$

Similar considerations hold if f is Lipschitz continuous in $[0,T]$ with Lipschitz constant L:

$$V_1[f](0,t) = \sum_k |f(t_k) - f(t_{k+1})| \leqslant \sum_k L(t_{k+1}-t_k) = LT\,.$$

Hence (3.6) implies that with probability one Brownian motion can not have a bounded derivative on any interval. In fact something much stronger is true. With probability one, Brownian motion is nowhere differentiable as a function of time (see [**13**] for details).

From Proposition 5.2, we see that $p=2$ is the border case. It is quite subtle. On one hand the statement (3.6) is true, yet if one considers a *specific* sequence of *shrinking* partitions $\Gamma^{(N)}$ (such that each successive partition contains the previous partition as a sub-partition) then

$$Q_N(T) := \sum_{\Gamma^{(N)}} |B(t_{k+1}^{(N)}) - B(t_k^{(N)})|^2 \to T \qquad \text{a.s.}\,.$$

Initially we will prove the following simpler statement.

THEOREM 5.3. *Let* $\Gamma^{(N)}$ *be a sequence of partitions of* $[0, T]$ *as in* (3.5) *with* $\lim_{N\to\infty} |\Gamma^{(N)}| \to 0$. *Then*

$$Q_N(T) := \sum_{\Gamma^{(N)}} |B(t_{k+1}^{(N)}) - B(t_k^{(N)})|^2 \underset{N\to\infty}{\longrightarrow} T \tag{3.7}$$

in $L^2(\Omega, \mathbb{P})$.

COROLLARY 5.4. *Under the conditions of the above theorem we have* $\lim_{N\to\infty} Q_N(T) = T$ *in probability.*

PROOF. We see that for any $\varepsilon > 0$

$$\mathbb{P}[\omega : |Z_N(\omega) - T| > \varepsilon] \leqslant \frac{\mathbb{E}[|Z_n(\omega) - T|^2]}{\varepsilon^2} \to 0 \qquad \text{as} \qquad |\Gamma^{(N)}| \to 0.$$

$\square$

PROOF OF THEOREM 5.3. Fix any sequence of partitions

$$\Gamma^{(N)} := \{\{t_k^{(N)}\} \ : \ 0 = t_1^{(N)} < t_2^{(N)} \cdots < t_n^{(N)} = T\},$$

of $[0, T]$ with $|\Gamma^{(N)}| \to 0$ as $N \to \infty$. Defining

$$Z_N := \sum_{k=1}^{N-1} [B(t_{k+1}^{(N)}) - B(t_k^{(N)})]^2,$$

we need to show that

$$\mathbb{E}[Z_N - T]^2.$$

We have,

$$\mathbb{E}[Z_N - T]^2 = \mathbb{E}[Z_N]^2 - 2T\mathbb{E}[Z_N] + T^2 = \mathbb{E}[Z_N]^2 - T^2.$$

Using the convenient notation $\Delta_k B := B(t_{k+1}^{(N)}) - B(t_k^{(N)})$ and $\Delta_k t^{(N)} := |t_{k+1}^{(N)} - t_k^{(N)}|$ we have that

$$\begin{aligned}
\mathbb{E}[Z_N]^2 &= \mathbb{E}[\sum_n (\Delta_n B)^2 \sum_k (\Delta_k B)^2] \\
&= \mathbb{E}[\sum_n (\Delta_n B)^4] + \mathbb{E}[\sum_{n\neq k} (\Delta_k B)^2 (\Delta_n B)^2] \\
&= 3\sum_n (\Delta_n t^{(N)})^2 + \sum_{n\neq k} (\Delta_k t^{(N)})(\Delta_n t^{(N)})
\end{aligned}$$

since $\mathbb{E}(\Delta_k B)^2 = \Delta_k t$ and $\mathbb{E}(\Delta_k B)^4 = (\Delta_k t^{(N)})^2$ because $\Delta_k B$ is a Gaussian random variable with mean zero and variance $\Delta_k t^{(N)}$.

The limit of the first term equals 0 as the maximum partition spacing goes to zero since

$$\sum_n (\Delta_n t^{(N)})^2 \leqslant 3 \cdot \sup(\Delta_n t^{(N)}) T$$

Returning to the remaining term

$$\begin{aligned}\sum_{n\neq k}(\Delta_k t^{(N)})(\Delta_n t^{(N)}) &= \sum_{k=1}^{N}\Delta_k t^{(N)}[\sum_{n=1}^{k-1}\Delta_n t^{(N)} + \sum_{k+1}^{N}\Delta_n t^{(N)}] \\ &= \sum_{k=1}^{N}\Delta_k t^{(N)}(T-\Delta_k t^{(N)}) \\ &= T\sum\Delta_k t^{(N)} - \sum(\Delta_k t^{(N)})^2 \\ &= T^2 - 0\end{aligned}$$

Summarizing, we have shown that

$$\mathbb{E}[Z_N - T]^2 \to 0 \qquad \text{as} \qquad N\to\infty$$

□

Corollary 5.5. *Under the conditions of Theorem 5.3, if* $\Gamma^{(N)}\subset\Gamma^{(N)}$ *we have* $\lim_{N\to\infty}Q_N(T) = T$ *almost surely.*

6. More Properties of Random Walks

We now return to the family of random walks constructed in Section 1. The collection of random walks $B^{(n)}(t)$ constructed in (3.1) have additional properties which are useful to identify and isolate as the general structures will be important for our development of stochastic calculus.

Fixing an $n\geqslant 0$, define $t_k = k2^{-n}$ for $k=0,\dots,2^n$. Then notice that for each such k, $B^{(n)}(t_k)$ is a Gaussian random variable since it is the sum of the mutually independent random variables $\xi_j^{(n)}$. Furthermore for any collection of $(t_{k_1},t_{k_2},\dots,t_{k_m})$ with $k_j\in\{1,\dots,2^n)\}$ we have that

$$\big(B^{(n)}(t_1),\cdots,B^{(n)}(t_m)\big)$$

is a multidimensional Gaussian vector.

Next notice that for $0\leqslant t_\ell < t_k\leqslant 1$ we have

$$\mathbb{E}\big[B_{t_k}^{(n)}\big|B_{t_\ell}^{(n)}\big] = \mathbb{E}\big[(B_{t_k}^{(n)}-B_{t_\ell}^{(n)}) + B_{t_\ell}^{(n)}\big|B_{t_\ell}^{(n)}\big] = \mathbb{E}\big[B_{t_k}^{(n)}-B_{t_\ell}^{(n)}\big] + B_{t_\ell}^{(n)} = B_{t_\ell}^{(n)} \tag{3.8}$$

since $B_{t_k}^{(n)}-B_{t_\ell}^{(n)}$ is independent of $B_{t_\ell}^{(n)}$ and $\mathbb{E}[B_{t_k}^{(n)}-B_{t_\ell}^{(n)}]=0$. We will choose to view this single fact as the result of two finer grain facts. The first being that the distribution of the walk at time t_k given the values $\{B_s^{(n)}: s<t_\ell\}$ is the same as the conditional distribution of the walk at time t_k given only $B_{t_\ell}^{(n)}$. In light of Definition 4.1, we can state this more formally by saying that for all functions f

$$\mathbb{E}\Big[f(B_{t_k}^{(n)})\Big|\mathcal{F}_{t_\ell}\Big] = \mathbb{E}\Big[f(B_{t_k}^{(n)})\Big|B_{t_\ell}^{(n)}\Big]$$

where $\mathcal{F}_t = \sigma(B_s^{(n)}: s\leqslant t)$. This property is called the Markov property which states that the distribution of the future depends only on the past through the present value of the process.

There is a stronger version of this property called the strong Markov property which states that one can in fact restart the process and restart it from the current (random) value and run it for the remaining amount of time and obtain the same answer. To state this more precisely let us introduce the process $X(t) = x + B^{(n)}(t)$ as the random walk starting from the point x and let $\mathbb{P}_x$ be the probability distribution induced on $C([0,1]\mathbb{R})$ by the trajectory of $X(t)$ for fixed initial x. Let $\mathbb{E}_x$ be the expected value associated to $\mathbb{P}_x$. Of course $\mathbb{P}_0$ is simply the random walk starting from 0 that we have been previously considering. Then the strong Markov property states that for any function f

$$\mathbb{E}_0 f(X_{t_k}) = \mathbb{E}_0 F(X_{t_k - t_\ell}, t_k) \qquad \text{where} \qquad F(x,t) = \mathbb{E}_x f(X_t)\,.$$

Neither of these Markov properties is solely enough to produce (3.8). We also need some fact about the mean of the process given the past. Again defining $\mathcal{F}_t = \sigma(B^{(n)}_s : s \leqslant t)$, we can rewrite (3.8) as

$$\mathbb{E}\Big[B^{(n)}_{t_k}\Big|\mathcal{F}_{t_\ell}\Big] = B^{(n)}_{t_\ell}$$

by using the Markov property. This equality is the principle fact that makes a process what is called a martingale.

We now revisit these ideas making more general definitions which abstract these properties so we can talk about and use them in broader contexts.

7. More Properties of General Stochastic Processes

Gaussian processes. We begin by giving the general definition of a Gaussian process of which Brownian motion is an example.

DEFINITION 7.1. $\{X_t\}$ *is a* Gaussian random process *if all finite dimensional distributions of X are Gaussian random variables.* I.e., *for all $t_1 < \dots t_k \in \mathcal{T}$ and $A_1, \dots, A_k \in \mathcal{B}$ (where here $\mathcal{B}$ represents Borel sets on the real line) we have that there exists R a positive definite symmetric $k \times k$ matrix and $m \in \mathbb{R}^k$ so that*

$$\mathbb{P}[X_{t_1} \in A_1 \text{ and } \dots \text{ and } X_{t_k} \in A_k] = \int_{A_1 \times \dots \times A_k} \frac{1}{(2\pi)^{k/2}\sqrt{\det R}} e^{-\frac{1}{2}(X-m)^T R^{-1}(X-m)}\,.$$

We have the associated definitions

$$\mu_t := \mathbb{E}[X_t]\,, \qquad R_{t,s} := \operatorname{cov}(X_t X_s) = \mathbb{E}[(X_t - \mu_t)\cdot(X_s - \mu_s)]\,.$$

EXAMPLE 7.2. *By definition, Brownian motion is a Gaussian process with mean vector $\mu_t = 0$ for all $t > 0$ and covariance matrix*

$$\begin{aligned} Cov(B_t, B_s) &= Cov(B_s + (B_t - B_s), B_s) = \mathbb{E}\left[B_s\right]^2 + \mathbb{E}\left[(B_t - B_s)B_s\right] \\ &= \mathbb{E}\left[B_s\right]^2 + \mathbb{E}\left[B_t - B_s\right]\mathbb{E}\left[B_s\right] = s\,. \end{aligned}$$

where we have assumed without loss of generality that $s < t$ and in the third identity we have used the independence of increments of Brownian motion. This shows that for general $t, s \geqslant 0$ we have

$$Cov(B_t, B_s) = \min\{t, s\}\,. \tag{3.9}$$

In fact, because the mean and covariance structure of a Gaussian process completely determine the properties of its marginals, if a Gaussian process has the same covariance and mean as a Brownian motion then it **is** a Brownian motion.

THEOREM 7.3. *A Brownian motion is a Gaussian process with zero mean function, and covariance function* $\min(t,s)$*. Conversely, a Gaussian process with zero mean function, and covariance function* $\min(t,s)$ *is a Brownian motion.*

PROOF. Example 7.2 proves the forward direction. To prove the reverse direction, assume that X_t is a Gaussian process with zero mean and $\operatorname{Cov}(X_t, X_s) = \min(t,s)$. Then the increments of the process, given by $(X_t, X_{t+s} - X_t)$ are Gaussian random variables with mean 0. The variance of the increments $X_{t+s} - X_t$ is given by

$$\operatorname{Var}(X_{t+s} - X_t, X_{t+s} - X_t) = \operatorname{Cov}(X_{t+s}, X_{t+s}) - 2\operatorname{Cov}(X_t, X_{t+s} - X_t) + \operatorname{Cov}(X_t, X_t) = (t+s) - 2t + t = s\,.$$

The independence of X_t and $X_{t+s} - X_t$ follows immediately by

$$\operatorname{Cov}(X_t, X_{t+s} - X_t) = \operatorname{Cov}(X_t, X_{t+s}) - \operatorname{Cov}(X_t, X_t) = t - t = 0\,.$$

$\square$

Martingales. In order to introduce the concept of a martingale, we first adapt the concept of σ-algebra to the framework of stochastic processes. In particular, in the case of a stochastic process we would like to encode the idea of **history** of a process: by observing a process up to a time $t > 0$ we have all the information on the behavior of the process before that time but none after it. Furthermore, as t increases we increase the amount of information we have on that process. This idea is the one that underlies the concept of *filtration*:

DEFINITION 7.4. *Given an indexing set T, a* filtration of σ-algebras *is a set of sigma algebras $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ such that for all $t_1 < \cdots < t_m \in \mathcal{T}$ we have*

$$\mathcal{F}_{t_1} \subset \cdots \subset \mathcal{F}_{t_m}\,.$$

We now define in which sense a filtration contains the information associated to a certain process

DEFINITION 7.5. *A stochastic process $\{X_t\}$ is* adapted *to a filtration $\{\mathcal{F}_t\}$ if its marginals are measurable with respect to the corresponding σ-algebras, i.e., if $\sigma(X_t) \subseteq \mathcal{F}_t$ for all $t \in \mathcal{T}$. In this case we say that the process is to the filtration $\{\mathcal{F}_t\}$.*

We also extend the concept of σ-algebras *generated* by a random variable to the case of a filtration. In this case, the filtration generated by a process $\{X_t\}$ is the smallest filtration containing enough information about $\{X_t\}$.

DEFINITION 7.6. *Let $\{X_t\}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. Then the filtration $\{\mathcal{F}_t^X\}$* generated *by X_t is given by*

$$\mathcal{F}_t^X = \sigma(X_s | 0 \leqslant s \leqslant t)\,,$$

which means the smallest σ-algebra with respect to which the random variable X_s is measurable, for all $s \in [0, t]$. Thus, $\mathcal{F}_t^X$ contains $\sigma(X_s)$ for all $0 \leqslant s \leqslant t$.

The canonical example of this is the filtration generated by a discrete random process (i.e. $\mathcal{T} = \mathbb{N}$):

EXAMPLE 7.7 (Example 1.1 continued). *The filtration generated by the N-coin flip process for $m < N$ is*

$$\mathcal{F}_m := \sigma(X_0, \ldots, X_m)\,.$$

Intuitively, we will think of $\{F_t^X\}$ as the **history** of $\{X_t\}$ up to time t, or the "information" about $\{X_t\}$ up to time t. Roughly speaking, an event A is in $\{F_t^X\}$ if its occurrence can be determined by knowing $\{X_s\}$ for all $s \in [0, t]$. For example, if B_t is a Brownian motion consider the event

$$A = \{\omega \in \Omega \; : \; \max_{t\in(0,1/2)} |B_s(\omega)| \leqslant 2\}\,.$$

It is clear that we have $A \in \mathcal{F}_{1/2}$ as the history of B_t up to time $t = 1/2$ determines whether A has occurres or not. However, we have that $A \notin \mathcal{F}_{1/3}$ as the process may not yet have reached 2 at time $t = 1/3$ but may do so before $t = 1/2$.

We now have all the tools to define the concept of a martingale:

DEFINITION 7.8. *$\{X_t\}$ is a* Martingale with respect to a filtration $\mathcal{F}_t$ *if for all $t > s$ we have*

i) *X_t is $\mathcal{F}_t$-measurable,*

ii) $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$,

iii) $\mathbb{E}[|X_t|] < \infty$.

Condition iii) in Def. 7.8 involves a conditional expectation with respect to the σ-algebra $\mathcal{F}_t$. Recall that $\mathbb{E}[X_{t+s}|\mathcal{F}_t]$ is an $\mathcal{F}_t$-measurable random variable which approximates X_{t+s} in a certain optimal way (and it is uniquely defined). Then Def. 7.8 states that, given the history of X_t up to time t, our best estimate of X_{t+s} is simply X_t , the value of $\{X_t\}$ at the present time t. In other words, a martingale is the equivalent in stochastic calculus of a straight line.

EXAMPLE 7.9. *Brownian motion is a martingale* WRT $\{\mathcal{F}_t^B\}$. *Indeed, we have that*

$$\mathbb{E}[B_{t+s}|\mathcal{F}_t] = \mathbb{E}[B_t + (B_{t+s} - B_t)|\mathcal{F}_t] = \mathbb{E}[B_t|\mathcal{F}_t] + \mathbb{E}[B_{t+s} - B_t|\mathcal{F}_t] = B_t + 0\,,$$

by the independence of increments property.

The above strategy can be extended to general functions $g(X)$, as the only property that was used is independence of the increments: because $g(X) \in \mathcal{F}^X$ this property implies that

$$\mathbb{E}\left[g(B_{t+s} - B_t)|\mathcal{F}_t^B\right] = \mathbb{E}\left[g(B_{t+s} - B_t)\right]\,. \tag{3.10}$$

EXAMPLE 7.10. *The process* $X_t := B_t^2 - t$ *is a martingale* WRT $\{\mathcal{F}_t^B\}$. *Indeed we have that the process is obviously measurable* WRT $\{\mathcal{F}_t^B\}$ *and that* $\mathbb{E}\left[|B_t^2|\right] = t < \infty$ *verifying i) and ii) from Def. 7.8. For iii) we have*

$$\begin{aligned}\mathbb{E}\left[B_{t+s}^2|\mathcal{F}_t^B\right] &= \mathbb{E}\left[(B_t + B_{t+s} - B_t)^2|\mathcal{F}_t^B\right]\\ &= \mathbb{E}\left[B_t^2|\mathcal{F}_t^B\right] - 2\mathbb{E}[B_t]\,\mathbb{E}\left[B_{t+s} - B_t|\mathcal{F}_t^B\right] + \mathbb{E}\left[(B_{t+s} - B_t)^2|\mathcal{F}_t^B\right] = B_t^2 + s\,.\end{aligned}$$

subtracting $t - s$ *on both sides of the above equation we obtain* $\mathbb{E}\left[B_{t+s}^2 - (t+s)|\mathcal{F}_t^B\right] = B_t^2 - t$.

EXAMPLE 7.11. *The process* $Y_t := \exp[\lambda B_t^2 - \lambda^2 t/2]$ *is a martingale* WRT $\{\mathcal{F}_t^B\}$. *Again, the process is obviously measurable* WRT $\{\mathcal{F}_t^B\}$ *and, computing the moment generating function of a Gaussian random variable, we have that* $\mathbb{E}[\exp(\lambda B_t)] = \exp\left(t\lambda^2/2\right) < \infty$, *verifying i) and ii) from Def. 7.8. For iii) we have*

$$\begin{aligned}\mathbb{E}\left[\exp(\lambda B_{t+s})\,|\mathcal{F}_t^B\right] &= \mathbb{E}\left[\exp(\lambda(B_t + B_{t+s} - B_t))\,|\mathcal{F}_t^B\right] = \exp(\lambda B_t)\,\mathbb{E}\left[\exp(\lambda(B_{t+s} - B_t))\,|\mathcal{F}_t^B\right]\\ &= \exp(\lambda B_t)\,\mathbb{E}[\exp(\lambda B_s)] = \exp\left(s\lambda^2/2\right)\,.\end{aligned}$$

multiplying by $\exp[(t-s)\lambda^2/2]$ *on both sides of the above equation we obtain*

$$\mathbb{E}\left[\exp\left(\lambda B_{t+s} - \lambda^2(t+s)/2\right)|\mathcal{F}_t^B\right] = \exp\left(\lambda B_t - \lambda^2 t/s\right)\,.$$

Markov processes. We now turn to the general idea of a Markov Process. As we have seen in the example above, this family of processes has the "memoryless" property, *i.e.*, their future depends on their past (their history, their *filtration*) only through their present (their state at the present time, or the σ-algebra generated by the random variable of the process at that time).

In the discrete time and countable sample space setting, this holds if given $t_1 < \cdots < t_m < t$, we have that the distribution of X_t given $(X_{t_1}, \ldots, X_{t_m})$ equals the distribution of X_t given (X_{t_m}). This is the case if for all $A \in \mathcal{B}(\mathbf{X})$ and $s_1, \ldots, s_m \in \mathbf{X}$ we have that

$$\mathbb{P}[X_t \in A|X_{t_1} = s_1 \ldots, X_{t_m} = s_m] = \mathbb{P}[X_t \in A|X_{t_m} = s_m]\,.$$

This property can be stated in more general terms as follows:

DEFINITION 7.12. *A random process* $\{X_t\}$ *is called* Markov *with respect to a filtration* $\{\mathcal{F}_t\}$ *when* X_t *is adapted to the filtration and, for any* $s > t$, X_s *is independent of* $\mathcal{F}_t$ *given* X_t.

The above definition can be restated in terms of Brownian motions as follows: For any set $A \in (\mathbf{X})$ we have

$$\mathbb{P}[B_t \in A|\mathcal{F}_s^B] = \mathbb{P}[B_t \in A|B_s] \qquad \text{a.s.}\,. \tag{3.11}$$

Remember that conditional probabilities with respect to a σ-algebra are really random variables in the way that a conditional expectation with respect to σ-algebra is a random variable. That is,

$$\mathbb{P}\left[B_t \in A|\mathcal{F}_s^B\right] = \mathbb{E}\left[\mathbb{1}_{B_t\in A}(\omega)|\mathcal{F}_s^B\right] \quad \text{and} \quad \mathbb{P}\left[B_t \in A|B_s\right] = \mathbb{E}\left[\mathbb{1}_{B_t\in A}(\omega)|\sigma(B_s)\right]\,.$$

EXAMPLE 7.13. *The fact that (3.11) holds can be shown directly by using characteristic functions. Indeed,to show that the distributions of the right and left hand side of (3.11) coincide it is enough to identify their characteristic functions. We compute*

$$\mathbb{E}\left[e^{i\vartheta B_t}|\mathcal{F}_s^B\right] = \mathbb{E}\left[e^{i\vartheta(B_s+B_t-B_s)}|\mathcal{F}_s^B\right] = e^{i\vartheta B_s}\mathbb{E}\left[e^{i\vartheta(B_t-B_s)}|\mathcal{F}_s^B\right] = e^{i\vartheta B_s}\mathbb{E}\left[e^{i\vartheta(B_t-B_s)}\right]\,,$$

and similarly

$$\mathbb{E}\left[e^{i\vartheta B_t}|B_s\right] = \mathbb{E}\left[e^{i\vartheta(B_s+B_t-B_s)}|B_s\right] = e^{i\vartheta B_s}\mathbb{E}\left[e^{i\vartheta(B_t-B_s)}|B_s\right] = e^{i\vartheta B_s}\mathbb{E}\left[e^{i\vartheta(B_t-B_s)}\right]\,.$$

Stopping times and Strong Markov property. We now introduce the concept of stopping time. As the name suggests, a stopping time is a time at which one can stop the process. The accent in this sentence should be put on **can**, and is to be intended in the following sense: if someone is observing the process as it evolves, and is given the instructions on when to stop, I can stop the process given his/her/their observations. In other words, the observer does not need future information to know if the event triggering the stop of the process has occurred or not. We now define this concept formally:

DEFINITION 7.14. *For a measurable space $(\Omega, \mathcal{F})$ and a filtration $\{\mathcal{F}_t\}$ with $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \in \mathcal{T}$, a random variable $\{\tau\}$ is a stopping time* WRT $\{\mathcal{F}_t\}$ *if $\{\omega \in \Omega \ : \ \tau \leqslant t\} \in \mathcal{F}_t$ for all $t \in \mathcal{T}$.*

Classical examples of stopping times are *hitting times* such as the one defined in the following example

EXAMPLE 7.15. *The random time $\tau_1 := \inf\{s > 0 \ : \ B_t \geqslant 1\}$ is a stopping time* WRT *the natural filtration of Brownian motion $\{\mathcal{F}_t^B\}$. Indeed, at any time t i can know if the event τ_1 has passed by looking at the past history of B_t. However, the random time $\tau_0 : +\sup\{s \in (0, t^*) \ : \ B_s = 0\}$ is NOT a stopping time* WRT $\{\mathcal{F}_t^B\}$ *for $t < t^*$, as before t^* we cannot know for sure if the process will reach 0 again.*

The *strong* Markov property introduced below is a generalization of the Markov property to stopping times (as opposed to fixed times in Def. 7.12). More specifically, we say that a stochastic process has the strong Markov property if its future after any stopping time depends on its past only through the present (*i.e.*, its state at the stopping time).

DEFINITION 7.16. *The stochastic process $\{X_t\}$ has the strong Markov property if for all finite stopping time τ one has*

$$\mathbb{P}\left[X_{\tau+s} \in A|\mathcal{F}_\tau\right] = \mathbb{P}\left[X_{\tau+s} \in A|X_\tau\right]\,,$$

where $\mathcal{F}_\tau := \{A \in \mathcal{F}_t \ : \ \{\tau \leqslant t\} \cap A \in \mathcal{F}_t\,, \forall t > 0\}$.

In the above definition, the σ-algebra $\mathcal{F}_\tau$ can be interpreted as "all the information we have on the process up to time τ".

THEOREM 7.17. *Brownian motion has the strong Markov property.*

We now use the above result to investigate some of the properties of Brownian motion:

EXAMPLE 7.18. *For any $t > 0$ define the maximum of Brownian motion in the interval $[0,t]$ as $M_t := \max_{s\in(0,t)} B_s$. Similarly, for any $m > 0$ we define the hitting time of m as $\tau_m := \inf\{s \in [0,t] \ : \ B_s \geqslant m\}$. Then, we write*

$$\begin{aligned}\mathbb{P}\left[M_t \geqslant m\right] &= \mathbb{P}\left[\tau_m \leqslant t\right] = \mathbb{P}\left[\tau_m \leqslant t, B_t \geqslant m\right] + \mathbb{P}\left[\tau_m \leqslant t, B_t < m\right] \\ &= \mathbb{P}\left[\tau_m \leqslant t, B_t - B_{\tau_m} \geqslant 0\right] + \mathbb{P}\left[\tau_m \leqslant t, B_t - B_{\tau_m} < m\right] .\end{aligned}$$

Using the strong Markov property of Brownian motion we have that $B_t - B_{\tau_m}$ is independent on $\mathcal{F}_{\tau_m}$ and is a Brownian motion. So by symmetry of Brownian motion we have that

$$\mathbb{P}\left[\tau_m \leqslant t, B_t - B_{\tau_m} \geqslant 0\right] = \mathbb{P}\left[\tau_m \leqslant t, B_t - B_{\tau_m} \leqslant m\right] = \mathbb{P}\left[B_t \geqslant m\right] ,$$

and we conclude that

$$\mathbb{P}\left[M_t \geqslant m\right] = 2\mathbb{P}\left[B_t \geqslant m\right] = \frac{2}{\sqrt{2\pi t}} \int_m^\infty e^{\frac{y^2}{2t}}\,\mathrm{d}x\,.$$

This argument is called the reflection principle *for Brownian motion. From the above argument one can also extract that*

$$\lim_{T\to\infty} \mathbb{P}\left[\tau_m < T\right] = 1\,,$$

i.e., that the hitting times of Brownian motion are almost surely finite.

From the above example we can also derive the following formula

EXAMPLE 7.19. *We will compute the probability density $\rho_{\tau_m}(s)$ of the hitting time of level m by the Brownian motion, defined by $\mathbb{P}\left[\tau_m \leqslant t\right] = \int_0^t \rho_{\tau_m}(s)\mathrm{d}s$. To do so we write*

$$\mathbb{P}\left[\tau_m \leqslant t\right] = \mathbb{P}\left[M_t \geqslant m\right] = 2\mathbb{P}\left[B_t \geqslant m\right] = \sqrt{\frac{2}{\pi t}} \int_m^\infty e^{-\frac{y^2}{2t}}\,\mathrm{d}y = \sqrt{2}\pi \int_{m/\sqrt{t}}^\infty e^{-u^2}\,\mathrm{d}u\,, \tag{3.12}$$

where in the last equality we made a change of variables $u = y/\sqrt{t}$. Now, differentiating (3.12) WRT *t we obtain, by Leibniz rule,*

$$\rho_{\tau_m}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\left[\tau_m \leqslant t\right] = \frac{|m|}{2\pi} t^{-3/2} e^{-\frac{m^2}{2t}}\,. \tag{3.13}$$

We immediately see from (3.13) that

$$\mathbb{E}\left[\tau_m\right] = \frac{|m|}{2\pi} \int_0^\infty s^{-1/2} e^{-\frac{m^2}{2s}}\,\mathrm{d}s = \infty\,.$$

EXAMPLE 7.20. *From the above computations we derive the distribution of zeros of Brownian motion in the interval $[a,b]$ for $0 < a < b$. We start by computing the desired quantity on the interval $[0,t]$ for an initial condition x which we assume* WLOG *to be $x < 0$:*

$$\begin{aligned}\mathbb{P}\left[B_s = 0 \text{ for } s \in [0,t] | B_0 = x\right] &= \mathbb{P}\left[\max_{s\in[0,t]} B_s \geqslant 0 | B_0 = x\right] = \mathbb{P}\left[\max_{s\in[0,t]} B_s \geqslant -x | B_0 = 0\right] \\ &= \mathbb{P}\left[\tau_{-x} \leqslant t\right] = \frac{|x|}{2\pi} \int_0^t s^{-3/2} e^{-\frac{x^2}{2s}}\,\mathrm{d}s\,. \end{aligned}\tag{3.14}$$

Since the above expression holds for all x we obtain the distribution of zeroes in the interval $[a,b]$ by integrating (3.14) over all possible x, weighted by the probability of reaching x at time a:

$$\begin{aligned}\mathbb{P}\left[B_s = 0 \text{ for } s \in [a,b] | B_0 = 0\right] &= \int_{-\infty}^\infty \mathbb{P}\left[B_s = 0 \text{ for } s \in [a, b-a] | B_a = x\right] \mathbb{P}\left[B_a \in \mathrm{d}x\right] \\ &\int_{-\infty}^\infty \frac{|x|}{2\pi} \int_0^{b-a} s^{-3/2} e^{-\frac{x^2}{2s}}\,\mathrm{d}s \sqrt{\frac{2}{\pi a}} e^{-\frac{x^2}{2a}}\,\mathrm{d}x = \frac{2}{\pi} \arccos\left(\sqrt{\frac{a}{b}}\right) .\end{aligned}$$

By taking the complement of the above we also obtain the probability that Brownian motion has no zeroes *in the interval* $[a,b]$*:*

$$\mathbb{P}\left[B_s \neq 0\,\forall s\in[a,b]\right] = 1 - \mathbb{P}\left[B_s = 0 \text{ for } s\in[a,b]\right] = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{a}{b}}\right).$$

The above result is referred to as the arcsine law *for Brownian motion.*

8. A glimpse of the connection with PDEs

The Gaussian

$$\rho(t,z) = \frac{e^{-\frac{z^2}{2\pi t}}}{\sqrt{2\pi t}}$$

is the fundamental solution to the heat equation. By direct calculation one sees that if $t>0$ then $\frac{\partial\rho}{\partial t} = \frac{1}{2}\frac{\partial^2\rho}{\partial x^2}$. There is a small problem at zero, namely ρ blows up. However, for any "nice" function $\phi(x)$ (smooth with compact support),

$$\lim_{t\to 0}\int \rho(t,x)\phi(x)dx = \phi(0).$$

This is the definition of the "delta function" $\delta(x)$. (If this is uncomfortable to you, look at [**18**].) Hence we see that $\rho(t,x)$ is the (weak) solution to

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \frac{1}{2}\frac{\partial^2\rho}{\partial x^2}\\ \rho(0,x) &= \delta(x)\end{aligned}$$

To see the connection to probability, we set $p(t,x,y) = \rho(t,x-y)$ and observe that for any function f we have

$$\mathbb{E}\{f(B_t)\big|B_0 = x\} = \int_{-\infty}^{\infty} f(y)p(t,x,y)dy$$

We will write $\mathbb{E}_x f(B_t)$ for $\mathbb{E}\{f(B_t)\big|B_0 = x\}$. Now notice that if $u(t,x) = \mathbb{E}_x f(B_t)$ then $u(t,x)$ solves

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\\ u(0,x) &= f(x)\end{aligned}$$

This is Kolmogorov Backward equation. We can also write it in terms of the transition density $p(t,x,y)$

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{1}{2}\frac{\partial^2 p}{\partial x^2}\\ p(0,x,y) &= \delta(x-y)\end{aligned}$$

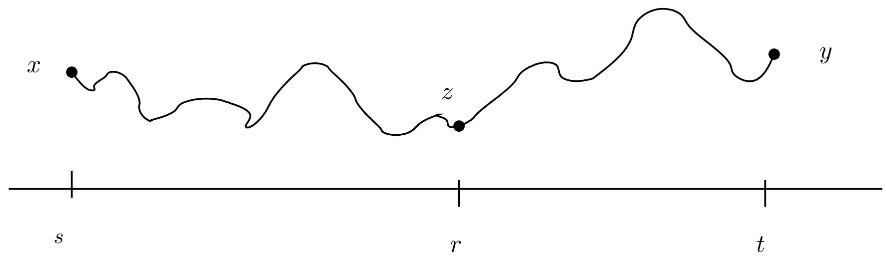
From this we see why it is called the "backwards" equation. It is a differential equation in the x variable. This is the "backwards" equation in $p(t,x,y)$ in that it gives the initial point. This begs a question. Yes, there is also a forward equation. It is written in terms of the forward variable y.

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{1}{2}\frac{\partial^2 p}{\partial y^2}\\ p(0,x,y) &= \delta(x-y)\end{aligned}$$

In this case it is identical to the backwards equation. In general it will not be.

We make one last observation: the $p(t,s,x,y) = p(t-s,x,y) = \rho(t-s,x-y)$ satisfy the Chapman-Kolmogorov equation (the semi-group property). Namely, for any $s<r<t$ and any x,y we have

$$p(s,t,x,y) = \int_{-\infty}^{\infty} p(s,r,x,z)p(r,t,z,y)dz$$



This also suggests the following form for the Kolmogorov forward equation. If we write an equation for $p(s,t,x,y)$ evolving in s and y, then we get an equation with a finial condition instead of an initial condition. Namely, for $s \leqslant t$

$$\frac{\partial p}{\partial s} = -\frac{1}{2}\frac{\partial^2 p}{\partial y^2}$$
$$\rho(t,t,x,y) = \delta(x-y)$$

Hence, we are solving backwards in time.

CHAPTER 4

Itô Integrals

1. Properties of the noise Suggested by Modeling

If we want to model a process X_t which was subject to random noise we might think of writing down a differential equation for X_t with a "noise" term regularly injecting randomness like the following:

$$\frac{dX_t}{\mathrm{d}t} = f(X_t) + g(X_t) \cdot (noise)_t$$

The way we have written it suggests the "noise" is generic and is shaped to the specific state of the system by the coefficient $g(X_t)$.

It is instructive to write this in integral form over the interval $[0, t]$

$$X_t = X_0 + \int_0^t f(X_s)\, \mathrm{d}s + \int_0^t g(X_s) \cdot (noise)_s \, \mathrm{d}s \tag{4.1}$$

it is reasonable to take the "noise" term to be a pure noise, independent of the structure of X_t and leave the "shaping" of the noise to a particular setting to the function $g(X_t)$. Since we want all moments of time to be the same it is reasonable to assume that distribution of "noise" is stationary in time. We would also like the noise at one moment to be independent of the noise at different moment. Since in particular both of these properties should hold when the function $g \equiv 1$ we consider that simplified case to gain insight.

Defining

$$V_t = \int_0^t (noise)_s \, ds$$

stationarity translates to the distribution of $V_{t+h} - V_t$ being independent of t. Independence translates to

$$V_{t_1} - V_0, V_{t_2} - V_{t_1}, \cdots, V_{t_{n+1}} - V_{t_n}$$

being a collection of mutually independent random variables for any collection of times

$$0 < t_1 < t_2 < \cdots < t_n \, .$$

Rewriting (4.1) with $g \equiv 1$ produces

$$X_t = X_0 + \int_t^t f(X_s)\, \mathrm{d}s + V_t$$

we see that if further decide that we would like to model processes X_t which are continuous in time we need to require that $t \mapsto V_t$ is almost surely a continuous process.

Clearly from our informal definition, $V_0 = 0$. Collecting all of the properties we desire of V_t:

i) $V_0 = 0$
ii) Stationary increments
iii) Independent increments
iv) $t \mapsto V_t$ is almost surely continuous.

Comparing this list with Theorem 3.6 we see that V_t must be a Brownian motion. We will chose it to be standard Brownian motion to fix a normalization.

Hence we are left to make sense of the integral equation

$$X_t = X_0 + \int_0^t f(X_s)dS + \int_0^t g(X_s)\frac{\mathrm{d}B_s}{\mathrm{d}s}\,\mathrm{d}s$$

Of course this leads to its own problems since we saw in Section 5 that B_s is nowhere differentiable. Formally canceling the "ds" maybe we can make sense of the integral

$$\int_0^t g(X_s)dB_s\,.$$

There is a well established classical theory of integrals in this form called "Riemann–Stieltjes" integration. We will briefly sketch this theory in the next section. However, we will see that even this theory is not applicable to the above integral. This will lead us to consider a new type of integration theory designed explicitly for random functions like Brownian motion. This named the Itô Integral after Kiyoshi Itô who developed the modern version though earlier version exist (notably in the work of Paley, Wiener and Zygmund).

2. Riemann-–Stieltjes Integral

Before we try to understand how to integrate against Brownian motion, we recall the classical Riemann—Stieltjes integration theory. Given two continuous functions f and g, we want to define

$$\int_0^T f(t)dg(t)\,. \tag{4.2}$$

We begin by considering a piecewise function ϕ function defined by

$$\phi(t) = \begin{cases} a_0 & \text{for } t \in [t_0, t_1] \\ a_k & \text{for } t \in (t_k, t_{k+1}], k = 1, \dots, n-1 \end{cases} \tag{4.3}$$

for some partition

$$0 = t_0 < t_1 < \cdot < t_{n-1} < t_n = t$$

and constants $a_k \in \mathbb{R}$. For such a function ϕ it is intuitively clear that

$$\int_0^T \phi(t)dg(t) = \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \phi(s)dg(s) = \sum_{k=0}^{n-1} a_k \int_{t_k}^{t_{k+1}} dg(s) = \sum_{k=0}^{n-1} a_k[g(t_{k+1}) - g(t_k)]\,. \tag{4.4}$$

because $\int_{t_k}^{t_{k+1}} dg(s) = g(t_{k+1}) - g(t_k)$ by the fundamental theorem of Calculus (since $\int_{t_k}^{t_{k+1}} dg(s) = \int_{t_k}^{t_{k+1}} g'(s)\,\mathrm{d}s$ if g is differentiable).

The basic idea of defining (4.2) is to approximate f by a sequence of step functions $\{\phi_n(t)\}$ each of the form given in (4.3) so that

$$\sup_{t\in[0,T]} |f(t) - \phi_n(t)| \to 0 \quad \text{as} \quad n \to \infty\,. \tag{4.5}$$

A natural choice of partition for the nth level is $t_k^{(n)} = Tk2^{-n}$ for $k = 0, \dots, 2^n$ and then define the nth approximating function by

$$\phi_n(t) = \begin{cases} f(T) & \text{if } t = T \\ f(t_k^{(n)}) & \text{if } t \in [t_k^{(n)}, t_{k+1}^{(n)}) \end{cases}$$

If f is continuous, it easy to see that (4.5) holds.

We then are left to show that there exists a constant α so that

$$|\int_0^T \phi^{(n)}(t)dg(t) - \alpha| \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

We would then define the integral $\int_0^T f(t)dg(t)$ to be equal to α. One of the keys to proving this convergence is a uniform bound on the approximating integrals $\int_0^T \phi^{(n)}(t)dg(t)$. Observe that

$$|\int_0^T \phi^{(n)}(t)dg(t)| \leqslant \|f\|_\infty \sum_{k=0}^{n-1} |g(t_k) - g(t_{k+1})| \leqslant \|f\|_\infty V_1[g](0,T)$$

where

$$\|f\|_\infty = \sum_{t\in[0,T]} |f(t)| \, .$$

This uniform bound implies that the $\int_0^T \phi^{(n)}(t)dg(t)$ say in a compact subset of $\mathbb{R}$. Hence there must be a limit point α of the sequence and a subsequence which converges to it. It is not then hard to show that this limit point is unique. That is to say that if any other subsequent converges it must also converge to α. We define the value of $\int_0^T f(t)dg(t)$ to be α. Hence it seems sufficient for g to have $V_1[g](0,T) < \infty$ if when g and f are continuous. It can also be shown to be necessary for a reasonable class of f.

It is further possible to show using essentially the same calculations that the limit α is independent of the sequence of partitions as long as the maximal spacing goes to zero and independent of the choice of point at which to evaluate the integrand f. In the above discussion we chose the left hand endpoint t_k of the interval $[t_k, t_{k+1}]$. However we were free to choose any point in the interval.

While the compactness argument above is a standard path in mathematics is often more satisfying to explicitly show that the $\{\phi^{(n)}\}$ are a Cauchy sequence by showing that for any $\varepsilon > 0$ there exists and N so that if $n, m > N$ then

$$\Big|\int_0^T \phi^{(m)}(t)dg(t) - \int_0^T \phi^{(n)}(t)dg(t)\Big| < \varepsilon$$

Since $\phi^{(m)} - \phi^{(n)}$ is again a step function of the form (4.3), the integral $\int_0^T [\phi^{(m)} - \phi^{(n)}](t)dg(t)$ is well defined given by a sum of the form (4.4). Hence we have

$$\Big|\int_0^T \phi^{(m)}(t)dg(t) - \int_0^T \phi^{(n)}(t)dg(t)\Big| = \Big|\int_0^T [\phi^{(m)} - \phi^{(n)}](t)dg(t)\Big| \leqslant \|\phi^{(m)} - \phi^{(n)}\|_\infty V_1[g](0,T)$$

Since f is continuous and the partition spacing is going to zero it is not hard so see that the $\{\phi^{(n)}\}$ from a Cauchy sequence under the $\|\cdot\|_\infty$ norm which completes the proof that integrals of the step functions form a Cauchy sequence.

3. A motivating example

We begin by considering the example

$$\int_0^T B_s \,\mathrm{d}B_s \tag{4.6}$$

where B is a standard Brownian motion. Since $V_1[B](0,T) = \infty$ almost surely we can not entirely follow the prescription of the Riemann-Stieltjes integral given in Section 2. However, it still seems reasonable to approximate the integrand B_s by a sequence of step functions of the form(4.3). However, since B is random, the a_k from (4.3) will have to be random variables.

Fixing a partition $0 = t_0 < t_1 < \cdots < t_N = T$, we define two different sequences of step function approximations of B. For $t \in [0, T]$, we define

$$\begin{aligned} \phi^N(t) = &\quad B(t_k) &&\text{if } t \in [t_k, t_{k+1}) \\ \hat{\phi}^N(t) = &\quad B(t_{k+1}) &&\text{if } t \in (t_k, t_{k+1}] \end{aligned}$$

Just as in the Riemann-Stieltjes setting (see (4.4)), for such step functions it is clear that one should define the respective integrals in the following manner:

$$\begin{aligned} \int_0^T \phi^N(t)\,\mathrm{d}B_s &= \sum_{i=0}^{N-1} B(t_k)[B(t_{k+1}) - B(t_k)] \\ \int_0^T \hat{\phi}^N(t)\,\mathrm{d}B_s &= \sum_{i=0}^{N-1} B(t_{k+1})[B(t_{k+1}) - B(t_k)] \end{aligned}$$

In the Riemann-Stieltjes setting, the two are the same. But in this case, the two have very different properties as the following calculation shows. Since $B(t_k)$ and $B(t_{k+1}) - B(t_k)$ are independent we have $\mathbb{E}B(t_k)[B(t_{k+1}) - B(t_k)] = \mathbb{E}\big[B(t_k)\big]\mathbb{E}[B(t_{k+1}) - B(t_k)] = 0$. So

$$\begin{aligned} \mathbb{E}\Big[\int_0^T \phi^N \mathrm{d}B_s\Big] &= \sum_{k=0}^{N-1} \mathbb{E}B(t_k)[B(t_{k+1}) - B(t_k)] \\ &= \sum_{k=0}^{N-1} \mathbb{E}\big[B(t_k)\big]\mathbb{E}[B(t_{k+1}) - B(t_k)] = 0 \end{aligned}$$

While since $\mathbb{E}[B(t_{k+1}) - B(t_k)]^2 = t_{k+1} - t_k$, we have

$$\begin{aligned} \mathbb{E}\Big[\int_0^T \hat{\phi}^N \mathrm{d}B_s\Big] =& \mathbb{E}\sum_{i=0}^{N-1} B(t_k)[B(t_{k+1}) - B(t_k)] + \mathbb{E}\sum_{k=0}^{N-1}[B(t_{k+1}) - B(t_k)]^2 \\ =& 0 + \sum_{k=0}^{N-1}\big[t_{k+1} - t_k\big] = T \end{aligned}$$

Hence, how we construct our step functions will be important in our analysis. The choice of the endpoint used in $\phi^N(t)$ leads to what is called the *Itô integral*. The choice used in $\hat{\phi}^N(t)$ is called the *Klimontovich Integral*. While if the midpoint is chosen, this leads to the *Stratonovich integral*. The question on which to use is a modeling question and is dependent on the problem being studied. We will see that it is possible to translate between all three in most cases. We will concentrate on the Itô integral since it has some nice additional properties which make the analysis attractive.

4. Itô integrals for a simple class of step functions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and B_t be a standard Brownian motion. Let $\mathcal{F}_t$ be a filtration on the probability space to which B_t is adapted. For example, one could have $\mathcal{F}_t = \sigma(B_s : s \leqslant t)$ be the filtration generated by the Brownian motion B_t.

DEFINITION 4.1. *$\phi(t, \omega)$ is an elementary stochastic process if there exists a collection of bounded, disjoint intervals $\{I_k\} = \{[t_0, t_1), [t_1, t_2), \ldots, [t_{N-1}, t_N)\}$ associated to a partition $0 \leqslant t_0 < t_1 < \cdots < t_N$ and a collection of random variables $\{\alpha_k : k = 0, \ldots, N\}$ so that*

$$\phi(t, \omega) = \sum_{k=0}^{N} \alpha_k(\omega)\mathbf{1}_{I_k}(t)\,,$$

the random variable $\alpha_k(\omega)$ is measurable with respect to $\mathcal{F}_{t_k}$, and $\mathbb{E}[|\alpha_k(\omega)|^2] < \infty$. We denote the space of all elementary functions as $\mathcal{S}_2$.

To be precise, stochastic integrals are defined on the class of *progressively measurable* processes, defined below:

DEFINITION 4.2. *A stochastic process $\{X_t\}_{t\geqslant 0}$ on $(\Omega, \mathcal{F}, \mathbb{P}, \{F_t\})$ is called progressively measurable, if for any $t \geqslant 0$, $X_t(\omega)$, viewed as a function of two variables (t, ω) is $\mathcal{B}_{[0,t]} \times \mathcal{F}_t$-measurable, where $\mathcal{B}_{[0,t]}$ is the Borel σ-algebra on $[0, t]$.*

FACT 1. *Every adapted right continuous with left limits ("cadlag") or left continuous with right limits process is progressively measurable. The reason to assume progressive measurability is to ensure that the expectation and the integral can be interchanged (by Fubini's theorem).*

FACT 2. *Any progressively measurable process $X = \{X_t\}_{t\in[0,T]}$ can be approximated by a sequence of simple processes $X^n = \{X^n_t\}_{t\in[0,T]} \in \mathcal{S}_2$ in the L_2-sense, that is*

$$\mathbb{E}\left[\int_0^T |X^n_t - X_t|^2 \, \mathrm{d}t\right] = \int_0^T \mathbb{E}\left[|X^n_t - X_t|^2\right] \mathrm{d}t \to 0 \quad \text{as } n \to \infty\,.$$

*The proof of this approximation can be found in [**14**].*
Because of the above result, we will be able to extend the notion of integral to adapted processes, and we restrict our attention to such processes for the rest of the chapter.

Next, we define a functional I which will be our integral operator. That is to say $I(\phi) = \int_0^\infty \phi \, \mathrm{d}B$. Just as for Riemann-Stieltjes integral, if ϕ is an elementary stochastic process, it is relatively clear what we should mean by $I(\phi)$, namely

DEFINITION 4.3. *The stochastic integral operator of an elementary stochastic process is given by*

$$I(\phi) := \sum_k \alpha_k [B(t_{k+1}) - B(t_k)]\,.$$

We first observe that I satisfies one of the standard properties of an integral in that it is a linear functional. In other words, if $\lambda \in \mathbf{R}$, and ϕ and ψ are elementary stochastic processes then

$$I(\lambda\phi) = \lambda I(\phi) \qquad \text{and} \qquad I(\psi + \phi) = I(\psi) + I(\phi) \tag{4.7}$$

Thanks to our requirement that α_k are measurable with respect to the filtration associated to the left endpoint of the interval $[t_k, t_{k+1})$ we have the following properties which will play a central role in what follows and should be compared to the calculations in Section 3.

LEMMA 4.4. *If ϕ is an elementary stochastic processes then*

$$\mathbb{E} I(\phi) = 0 \qquad \textit{(mean zero)}$$

$$\mathbb{E}\big[I(\phi)^2\big] = \int_0^\infty \mathbb{E}|\phi(t)|^2 \, \mathrm{d}s \qquad \textit{(Itô Isometry)}$$

REMARK 4.5. *An isometry is a map between two spaces which preserves distance (i.e. the norm). If we consider*

$$I(\mathcal{S}_2) = \big\{I(\phi) : \phi \in \mathcal{S}_2\big\}$$

then according to Lemma 4.4 the map $I(\phi) \mapsto \phi$ is an isometry between the space of random variables

$$L^2\big(I(\mathcal{S}_2), \mathbb{P}\big) = \big\{X \in I(\mathcal{S}_2) : \|X\| = \sqrt{\mathbb{E}(X^2)} < \infty\big\}$$

and the space of elementary stochastic processes $L^2(\mathcal{S}_2, \mathbb{P}[d\omega] \times \mathrm{d}t]$ equipped with the norm

$$\|\phi\| = \Big(\int_0^\infty \mathbb{E}\phi^2(t, \omega) \, \mathrm{d}t\Big)^{\frac{1}{2}}$$

Proof of Lemma 4.4. We begin by showing $I(\phi)$ is mean zero.

$$\begin{aligned}\mathbb{E}[I(\phi)] &= \sum_k \mathbb{E}[\alpha_k(B(t_{k+1}) - B(t_k))] \\ &= \sum_k \mathbb{E}[\mathbb{E}[\alpha_k(B(t_{k+1}) - B(t_k))]|\mathcal{F}_{t_k}] \\ &= \sum_k \mathbb{E}[\alpha_k \mathbb{E}[B(t_{k+1}) - B(t_k)|\mathcal{F}_{t_k}]] = 0\end{aligned}$$

Turning to the Itô isometry

$$\begin{aligned}\mathbb{E}[I(\phi)^2] =& \Big[\mathbb{E}\big(\sum_k \alpha_k(B(t_{k+1}) - B(t_k))\big)\Big] \cdot \Big[\mathbb{E}\big(\sum_j \alpha_j(B(t_{j+1}) - B(t_j))\big)\Big] \\ =& \sum_{j,k} \mathbb{E}[\alpha_k\alpha_j[B(t_{k+1}) - B(t_k)][B(t_{j+1}) - B(t_j)]] \\ =& 2\sum_{j<k} \mathbb{E}[\alpha_k\alpha_j[B(t_{k+1}) - B(t_k)][B(t_{j+1}) - B(t_j)]] \\ &+ \sum_k \mathbb{E}[\alpha_k^2[B(t_{k+1}) - B(t_k)]^2]\end{aligned}$$

Next, we examine each component separately: Recall for $t_{k+1} \leqslant t_j$

$$\begin{aligned}&\mathbb{E}[\alpha_k\alpha_j[B(t_{k+1}) - B(t_k)][B(t_{j+1}) - B(t_j)]] \\ &\quad = \mathbb{E}[\mathbb{E}[\alpha_k\alpha_j[B(t_{k+1}) - B(t_k)][B(t_{j+1}) - B(t_j)]|\mathcal{F}_{t_j}]] \\ &\quad = \mathbb{E}[\alpha_k\alpha_j[B(t_{k+1}) - B(t_k)]\mathbb{E}[[B(t_{j+1}) - B(t_j)]|\mathcal{F}_{t_j}]] \\ &\quad = 0\end{aligned}$$

since $\mathbb{E}[[B(t_{j+1}) - B(t_j)]|\mathcal{F}_{t_j}] = 0$. Similarly

$$\begin{aligned}\sum_k \mathbb{E}[\alpha_k^2[B(t_{k+1}) - B(t_k)]^2] &= \sum_k \mathbb{E}[\mathbb{E}[\alpha_k^2[B(t_{k+1}) - B(t_k)]^2|\mathcal{F}_{t_k}]] \\ &= \sum_k \mathbb{E}[\alpha_k^2(t_{k+1} - t_k)]\end{aligned}$$

Hence, we have:

$$\mathbb{E}[I(\phi)^2] = 0 + \sum_k \mathbb{E}[\alpha_k^2](t_{k+1} - t_k) = \int \mathbb{E}[\phi^2(s)]\,\mathrm{d}s$$

□

So far we have just defined the Itô Integral on the whole positive half line $[0, \infty)$. For any $0 \leqslant s < t \leqslant \infty$, we make the following definition

$$\int_s^t \phi_r \,\mathrm{d}B_r = I(\phi\mathbf{1}_{[s,t)})$$

We can now talk about the stochastic process

$$M_t = I(\phi\mathbf{1}_{[0,t)}) = \int_0^t \phi_s dB_s \tag{4.8}$$

associated to a given elementary stochastic process ϕ_t, where the last two expressions are just different notation for the same object.

We now state a few simple consequences of our definitions.

LEMMA 4.6. *Let $\phi \in \mathcal{S}_2$ and $0 < s < t$ then*

$$\int_0^t \phi_r \, \mathrm{d}B_r = \int_0^s \phi_r \, \mathrm{d}B_r + \int_s^t \phi_r \, \mathrm{d}B_r$$

and

$$M_t = \int_0^t \phi_s \, \mathrm{d}B_s$$

is measurable with respect to $\mathcal{F}_t$.

PROOF OF LEMMA 4.6. Clearly M_t is measurable with respect to $\mathcal{F}_t$ since $\{\phi_s : s \leqslant t\}$ are by assumption and the construction of the integral only uses the information from $\{B_s : s \leqslant t\}$. The first property follows from $\phi \mathbf{1}_{[0,t]} = \phi \mathbf{1}_{[0,s)} + \phi \mathbf{1}_{[s,t)}$ and hence

$$I(\phi \mathbf{1}_{[0,t]}) = I(\phi \mathbf{1}_{[0,s)}) + I(\phi \mathbf{1}_{[s,t)})$$

by (4.7). □

LEMMA 4.7. *Let M_t be as in* (4.8) *for an elementary process ϕ and a Brownian motion B_t both adapted to a filtration $\{\mathcal{F}_t : t \geqslant 0\}$. Then M_t is a martingale with respect to the filtration $\mathcal{F}_t$.*

PROOF OF LEMMA 4.7. Looking at Definition 7.8, there are three conditions we need to verify. The measurability is contained in Lemma 4.6. The fact that $\mathbb{E}[M_t^2] < \infty$ follows from the Itô isometry since

$$\mathbb{E}[M_t^2] = \int_0^t \mathbb{E}|\phi_s|^2 \, \mathrm{d}s \leqslant \int_0^\infty \mathbb{E}|\phi_s|^2 \, \mathrm{d}s$$

because the last integral is assumed to be finite in the definition of an elementary stochastic process.

All that remains is to verify that for $s < t$.

$$\mathbb{E}[M_t(\phi) - M_s(\phi) | \mathcal{F}_s] = 0 \, .$$

There are a few cases. Taking one case, say s and t are in the disjoint intervals $[t_k, t_{k+1})$ and $[t_j, t_{j+1})$, respectively. We have that

$$M_t(\phi) - M_s(\phi) = \alpha_k [B(t_{k+1}) - B_s] + \Big(\sum_{n=k+1}^{j-1} \alpha_n [B(t_{n+1}) - B(t_n)] \Big) + \alpha_j [B_t - B(t_j)]$$

Next, take repeated expectations with respect to the filtrations $\{\mathcal{F}_a\}$, where $a \in \{t_j, t_{j-1}, \dots, t_{k+1}, s\}$. This would then imply that for each a

$$\begin{aligned} \mathbb{E}[\alpha_a [B(t_{a+1}) - B(t_a)] | \mathcal{F}_a] &= \alpha_a \mathbb{E}[B(t_{a+1}) - B(t_a) | \mathcal{F}_a] \\ &= \alpha_a \mathbb{E}[B(t_{a+1}) - B(t_a)] \\ &= 0 \end{aligned}$$

Hence, $\mathbb{E}[M_t(\phi) - M_s(\phi) | \mathcal{F}_s] = 0$. The other cases can be done similarly. And the conclusion immediately follows. □

LEMMA 4.8. *In the same setting as Lemma 4.7, M_t is a continuous stochastic process. (That is to say, with probability one the map $t \mapsto M_t$ is continuous for all $t \in [0, \infty)$).*

PROOF OF LEMMA 4.8. We begin by noticing that if $\phi(t, \omega)$ is a simple process with

$$\phi(t, \omega) = \sum_{k=1}^{N-1} \alpha_k(\omega) \mathbf{1}_{[t_k, t_{k+1})}(t)$$

then if $t \in (t_{k*}, t_{k*+1})$ then

$$M_t(\phi, \omega) = \sum_{k=1}^{k^*-1} \alpha_k(\omega)[B(t_{k+1}, \omega) - B(t_k, \omega)] + \alpha_{k*}(\omega)[B(t, \omega) - B(t_{k*}, \omega)]$$

Hence it is clear that $M_t(\phi, \omega)$ is continuous if ϕ is a simple function since the Brownian motion B_t is continuous. $\square$

5. Extension to the Closure of Elementary Processes

We will denote by $\bar{\mathcal{S}}_2$ the closure in $L^2(\mathbb{P} \times \mathrm{d}t)$ of the square integrable elementary processes $\mathcal{S}_2$. Namely,

$$\bar{\mathcal{S}}_2 = \Big\{ \text{all stochastic process } f(t, \omega) : \text{there exist a sequence } \phi_n(t, \omega) \in \mathcal{S}_2 \\ \text{so that } \int_0^\infty \mathbb{E}(f(t) - \phi_n(t))^2 \, \mathrm{d}t \to 0 \text{ as } n \to \infty \Big\}$$

Also recall that we define

$$L^2(\Omega, \mathbb{P}) := \Big\{ f : \Omega \to \mathbb{R} : \int_\Omega f(\omega)^2 \, \mathbb{P}[d\omega] < \infty \Big\} \\ = \{\text{Random Variables } X \text{ with } \mathbb{E}(X^2) < \infty\}$$

and

$$L^2(\Omega \times [0, \infty), \mathbb{P} \times \mathrm{d}t) := \Big\{ f : \Omega \times [0, \infty) : \int_0^\infty \int_\Omega f(t, \omega)^2 \, \mathbb{P}[d\omega] \, \mathrm{d}t < \infty \Big\} \\ = \Big\{ \text{Stochastic Processes } X_t \text{ with } \int_0^\infty \mathbb{E}(X_t^2) \, \mathrm{d}t < \infty \Big\}$$

In general, in the interest of brevity we will write $L^2(\Omega)$ for the first and $L^2(\Omega \times [0, \infty))$ for the second space above. We will occasionally write $\|X\|_{L^2(\Omega)}$ for $\sqrt{\mathbb{E}(X^2)}$ and $\|X\|_{L^2(\Omega \times [0,\infty))}$ for $(\int_0^\infty \mathbb{E}(X_t^2) \, \mathrm{d}t)^{\frac{1}{2}}$ though we will simply write $\| \cdot \|_{L^2}$ when the context is clear.

Recalling that our definition of $\mathcal{S}_2$ and hence $\bar{\mathcal{S}}_2$ had a filtration $\mathcal{F}_t$ in the background, we define $L^2_{\mathrm{ad}}(\Omega \times [0, \infty))$ to be the subset of $L^2(\Omega \times [0, \infty))$ in which all of the stochastic processes are adapted to the filtration $\mathcal{F}_t$. We have the following simple observation

Lemma 5.1. *The closure of the space of elementary processes is contained in the space of square integrable, adapted processes, i.e.,* $\bar{\mathcal{S}}_2 \subset L^2_{\mathrm{ad}}(\Omega \times [0, \infty))$.

Proof. The adaptedness follows from the adaptedness of $\mathcal{S}_2$. The fact that elements in $\bar{\mathcal{S}}_2$ are square integrable follows from the following calculation. Fixing an $X \in \bar{\mathcal{S}}_2$ and a sequence $\phi_n \in \bar{\mathcal{S}}_2$ so that $\|\phi_n - X\| \to 0$ as $n \to \infty$ we fix an n so that $\|\phi_n - X\| \leqslant 1$. Then since for any real numbers a and b and $p \geqslant 1$ one has $|a + b|^2 \leqslant 2^{p-1}|a|^2 + 2^{p-1}|b|^p$ we see that

$$\|f\|_{L^2}^2 = \|f_n + (f - \phi_n)\|_{L^2}^2 \leqslant 2\|\phi_n\|_{L^2}^2 + 2\|f - \phi_n\|_{L^2}^2 \leqslant 2(\|\phi_n\|_{L^2}^2 + 1) < \infty$$

where the term on the far right is finite since for every n, $\phi_n \in \mathcal{S}_2$ (observe that $\|\phi_n\|_{L^2}^2 = \sum \mathbb{E}(\alpha_k^2)$). $\square$

In fact, $\bar{\mathcal{S}}_2$ is *exactly* $L^2_{\mathrm{ad}}(\Omega \times [0, \infty))$,

Theorem 5.2. $\bar{\mathcal{S}}_2 = L^2_{\mathrm{ad}}(\Omega \times [0, \infty))$

Proof. $\bar{\mathcal{S}}_2 \subset L^2_{\mathrm{ad}}(\Omega \times [0, \infty))$ was just proven in Lemma 5.1. For the other direction, see the proof of [**14**, Theorem 3.1.5] or [**11**] $\square$

We are now ready to state the main theorem of this section. This result shows that given a Cauchy sequence of elementary stochastic processes $\{\phi_n\} \in \mathcal{S}_2$ converging in $L^2(\Omega, [0,\infty))$ to a process $f \in \bar{\mathcal{S}}_2$, there exists a random variable $X \in L^2(\Omega)$ to which the stochastic integrals $I(\phi_n)$ converge in $L^2(\Omega)$. We will then define this variable X to be the stochastic integral $I(f) = \int_0^\infty f_s \,\mathrm{d}B_s$. The concept of the construction is summarized in the following scheme:

$$\begin{tikzcd}
\{\phi_n(t)\}_{t\in\mathcal{T}} \in & \mathcal{S}^2 \arrow[r, leftrightarrow, "I(\cdot)"] \arrow[d, "\int_0^\infty \mathbb{E}\left[(\phi_n - f)^2\right]\mathrm{d}t \to 0"', "n\to\infty"] & I(S^2) \arrow[d, "n\to\infty"', "\mathbb{E}\left[(I(\phi_n) - X)^2\right] \to 0"] & \ni \int \phi_n \,\mathrm{d}B_s \\
\{f(t)\}_{t\in\mathcal{T}} \in & \bar{\mathcal{S}}^2 \arrow[r, leftrightarrow, dashed, "I(\cdot)"] & L^2(\Omega) & \ni X =: \int f \,\mathrm{d}B_s
\end{tikzcd}$$

THEOREM 5.3. *For every $f \in \bar{\mathcal{S}}_2$, there exists a random variable $X \in L^2(\Omega)$ so that if $\{\phi_n : n = 1, \ldots, \infty\}$ is a sequence of elementary stochastic processes (i.e. elements of $\mathcal{S}_2$) converging to the stochastic process f in $L^2(\Omega \times [0,\infty)$ then $I(\phi_n)$ converges to X in $L^2(\Omega)$.*

DEFINITION 5.4. *For any $f \in \bar{\mathcal{S}}_2$, we define the Itô integral*

$$I(f) = \int_0^\infty f_t \,\mathrm{d}B_t$$

to be the random variable $X \in L^2(\omega)$ given by Theorem 5.3.

PROOF OF THEOREM 5.3. We start by showing that $I(\phi_n)$ is a Cauchy sequence in $L^2(\Omega)$. By the linearity of the map I (see (4.7)), $I(\phi_n) - I(\phi_m) = I(\phi_n - \phi_m)$. Hence by the Itô isometry for the elementary stochastic processes (see lemma 4.4), we have that

$$\mathbb{E}\big[(I(\phi_n) - I(\phi_m))^2\big] = \mathbb{E}\big[I(\phi_n - \phi_m)^2\big] = \int_0^\infty \mathbb{E}\left[(\phi_n(t) - \phi_m(t))^2\right] \,\mathrm{d}t \tag{4.9}$$

Since the sequence ϕ_n converges to f in $L^2(\Omega \times [0,\infty))$, we know that it is a Cauchy sequence in $L^2(\Omega \times [0,\infty))$. By the above calculations we hence have that $\{I(\phi_n)\}$ is a Cauchy sequence in $L^2(\Omega)$. It is a classical fact from real analysis that this space is complete which implies that every Cauchy sequence converges to a point in the same space. Let $X \in L^2(\Omega)$ denote the limit.

To see that X does not depend on the sequence $\{\phi_n\}$, let $\{\tilde{\phi}_n\}$ be another sequence converging to f. The same reasoning as above ensures the existence of a $\tilde{X} \in L^2(\Omega)$ so that $I(\tilde{\phi}_n) \to \tilde{X}$ in $L^2(\Omega)$. On the other hand

$$\begin{aligned}
\mathbb{E}\left[(I(\phi_n) - I(\tilde{\phi}_n))^2\right] &= \mathbb{E}\left[(I(\phi_n - \tilde{\phi}_n))^2\right] = \int_0^\infty \mathbb{E}\left[(\phi_n(t) - \tilde{\phi}_m(t))^2\right] \,\mathrm{d}t \\
&\leqslant 2\int_0^\infty \mathbb{E}\left[(\phi_n(t) - f)^2\right] \,\mathrm{d}t + 2\int_0^\infty \mathbb{E}\left[(f - \tilde{\phi}_m(t))^2\right] \,\mathrm{d}t
\end{aligned}$$

where in the inequality we have again used the fact that $(\phi_n - \tilde{\phi}_m)^2 = [(\phi_n - f) + (f - \tilde{\phi}_m)]^2 \leqslant 2(\phi_n - f)^2 + 2(f - \tilde{\phi}_n)^2$. Since both ϕ_n and $\tilde{\phi}_n$ converge to f in $L^2(\Omega \times [0,\infty))$ we have that the last two terms on the right-hand side go to 0. This in turn implies that $\mathbb{E}(X - \tilde{X})^2 = 0$ and that the two random variables are the same. $\square$

Remark 5.5. *While the above construction might seem like magic since it is so soft, it is an application of a general principle in mathematics. If one can define a linear map on a dense subset of elements in a space in such a way that the map is an isometry then the map can be uniquely extend to the whole space. This approach is beautifully presented in our current context in* [**10**] *using the following lemma*

Lemma 5.6. *(Extension Theorem) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces. Let $\mathcal{B}_0 \subset \mathcal{B}_1$ be a linear space. If $L : \mathcal{B}_0 \to \mathcal{B}_2$ is defined for all $b \in \mathcal{B}_0$ and $|Lb|_{\mathcal{B}_2} = |Lb|_{\mathcal{B}_1}$, $\forall b \in \mathcal{B}_0$. Then there exists a unique representation of L to $\overline{\mathcal{B}}_0$ (closure of $\mathcal{B}_0$) called $\overline{L}$ with $Lb = \overline{L}b$, $\forall b \in \mathcal{B}_0$.*

Example 5.7. *We use the above theorem to show that*

$$I_T(B_s) = \int_0^T B_s \,\mathrm{d}B_s = \frac{1}{2}(B_T^2 - T)\,.$$

To do so, we show that the sequence $\phi_n = \sum_{j=1}^N \mathbb{1}_{[t_j^{(n)}, t_{j+1}^{(n)})} B_{t_j^n}$ with $\Delta t_j^n = t_{j+1}^{(n)} - t_j^{(n)} \to 0$ converges in $L^2(\Omega, [0,T))$ to $\{B_t\}$. Indeed we have that

$$\begin{aligned}\int_0^T \mathbb{E}\left[|\phi_n - B_s|^2\right] \mathrm{d}t &= \sum_{j=1}^N \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (B_{t_j^{(n)}} - B_s)^2 \,\mathrm{d}s = \sum_{j=1}^N \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (t_{j+1}^{(n)} - s)\mathrm{d}s \\ &= \frac{1}{2} \sum_{j=1}^N (t_{j+1}^{(n)} - t_j^{(n)})^2 \to 0\,.\end{aligned}$$

Then, by Theorem 5.3 we have that

$$\int_0^T B_s \,\mathrm{d}B_s = \lim_{n\to\infty} I(\phi_n) = \lim_{n\to\infty} \sum_j B_{t_j^{(n)}} \Delta B_j^n\,.$$

Now, writing $\Delta B_j^n := B_{t_j^{(n)}} - B_{t_{j+1}^{(n)}}$ we have

$$\Delta(B_j^2) := B_{t_j^{(n)}}^2 - B_{t_{j+1}^{(n)}}^2 = (B_{t_j^{(n)}} - B_{t_{j+1}^{(n)}})^2 + 2B_{t_j^{(n)}}(B_{t_j^{(n)}} - B_{t_{j+1}^{(n)}}) = (\Delta B_j^n)^2 + 2B_{t_j^{(n)}} \Delta B_j^n$$

and therefore

$$\sum_j B_{t_j^{(n)}} \Delta B_j^n = \frac{1}{2}\left(\sum_j \Delta(B_j^2) - \sum_j (\Delta B_j^n)^2\right) = \frac{1}{2}\left(B_T^2 - \sum_j (\Delta B_j^n)^2\right).$$

The term on the RHS *converges to $(B_T^2 - T)/2$ in $L^2(\Omega)$, which therefore corresponds to $\int_0^T B_s \,\mathrm{d}B_s$.*

6. Properties of Itô integrals

Proposition 6.1. *Let $f, g \in L^2_{\text{ad}}(\Omega, [0,\infty))$, $\lambda \in \mathbb{R}$, then*

i) *Linearity:*

$$\int_0^\infty (\lambda f_s + g_s)\mathrm{d}B_s = \lambda \int_0^\infty f_s \,\mathrm{d}B_s + \int_0^\infty g_s \,\mathrm{d}B_s\,, \tag{4.10}$$

ii) *Separability: for all $S > 0$,*

$$\int_0^\infty f_s \,\mathrm{d}B_s = \int_0^S f_s \,\mathrm{d}B_s + \int_S^\infty f_s \,\mathrm{d}B_s\,,$$

iii) *Mean 0:*

$$\mathbb{E}\left[\int_0^\infty f_s \,\mathrm{d}B_s\right] = 0\,, \tag{4.11}$$

iv) *Itô isometry:*

$$\mathbb{E}\left[\left(\int_0^\infty f_s \,\mathrm{d}B_s\right)^2\right] = \int_0^\infty \mathbb{E}\left[f_s^2\right]\mathrm{d}t\,. \tag{4.12}$$

PROOF. For the proof of points i) and ii) we refer to [**9**].
To prove iv), for $f \in L^2_{\mathrm{ad}}(\Omega, [0,\infty)$, let $\{\phi_n\} \in \mathcal{S}_2$ be a sequence of elementary stochastic processes converging to f in $L^2_{\mathrm{ad}}(\Omega, [0,\infty)$. By Theorem 5.3, there exists $X \in L_2(\Omega)$ with $\mathbb{E}\left[(X - I(\phi_n))^2\right] \to 0$ as $n \to \infty$. Since $\mathbb{E}\left[X^2\right], \mathbb{E}\left[I(\phi_n)^2\right] < \infty$ using Cauchy-Schwartz (or Hölder) inequality we write

$$\begin{aligned}
|\mathbb{E}\left[I(\phi_n)^2\right] - \mathbb{E}\left[X^2\right]| &= |\mathbb{E}\left[(X - I(\phi_n))(X + I(\phi_n))\right]| \\
&\leqslant |\mathbb{E}\left[(X - I(\phi_n))I(\phi_n)\right] + \mathbb{E}\left[(X - I(\phi_n))X\right]| \\
&\leqslant \sqrt{\mathbb{E}\left[X^2\right] + \mathbb{E}\left[I(\phi_n)^2\right]}\sqrt{\mathbb{E}\left[(X - I(\phi_n))^2\right]}\,,
\end{aligned}$$

with the term on the right hand side $\sqrt{\mathbb{E}\left[(X - I(\phi_n))^2\right]} = \|X - I(\phi_n)\|_{L^2(\omega)} \to 0$ as $n \to \infty$. This implies that in the limit $n \to \infty$ we have

$$\mathbb{E}\left[I(\phi_n)^2\right] \to \mathbb{E}\left[X^2\right]\,.$$

At the same time,

$$\mathbb{E}I(\phi_n)^2 = \int_0^\infty \mathbb{E}(\phi_n(t))^2 \,\mathrm{d}t \to \int_0^\infty \mathbb{E}(f(t)^2)\,\mathrm{d}t\,.$$

Combining these facts produces

$$\mathbb{E}(X^2) = \int_0^\infty \mathbb{E}(f(t)^2)\,\mathrm{d}t$$

as desired. The exact same logic produces $\mathbb{E}X = 0$. □

7. A continuous in time version of the the Itô integral

In this section, we consider the

DEFINITION 7.1. *For any $f \in L_{\mathrm{ad}}(\Omega \times [0,\infty))$ and any $t \geqslant 0$, we define the* Itô integral process *$\{I_t(f)\}$ as*

$$I_t(f) = \int_0^t f_s(\omega)\mathrm{d}B_s(\omega) := I(f\mathbf{1}_{(0,t]})\,.$$

Note that the process introduced above is well defined and adapted to $\mathcal{F}_t$. In fact since

$$\mathbb{E}I_t(f)^2 = \int_0^t \mathbb{E}f_t^2\,\mathrm{d}t \leqslant \int_0^\infty \mathbb{E}f_t^2\,\mathrm{d}t < \infty$$

we see that $I_t(f)$ is an adapted stochastic process whose second moment is uniformly bounded in time. It is not immediately clear that we can fix the realization ω (which in turn fixes the realization of f and W) and then change the time t in $I_t(f)(\omega) = \int_0^t f_s(\omega)\mathrm{d}B_s(\omega)$. We built the integral as some limit at each time t. Changing the time t requires us to repeat the limiting process. This would be fine except that we built the Itô integral through an L^2-limit. Hence at each time it is only defined up to sets of probability zero. If we only want to define I_t for some countable sequence of time then this still would not be a problem because the countable union of measure zero sets is still measure zero. However, we would like to define $I_t(f)$ for all $t \in [0,T]$. This is a problem and more work is required.

THEOREM 7.2. *Let $f \in L^2_{\mathrm{ad}}(\Omega, [0\infty))$, then there exists a continuous version of $I_t(f)$, i.e., there exists a t-continuous stochastic process $\{J_t\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for all $t \in \mathcal{T}$*

$$\mathbb{P}\left[J_t = I_t(f)\right] = 1\,.$$

To prove this result, we state a very useful theorem which we will prove later.

THEOREM 7.3 (Doob's Martingale Inequality). *Let M_t be a continuous-time martingale with respect to the filtration $\mathcal{F}_t$. Then*

$$\mathbf{P}\Big(\sup_{0\leqslant t\leqslant T}|M_t|\geqslant\lambda\Big)\leqslant\frac{\mathbb{E}[|M_T|^p]}{\lambda^p}$$

where $\lambda\in\mathbf{R}^+$, and $p\geqslant 1$.

PROOF OF THEOREM 7.2. By Lemma 4.8 we know that $I_t(\phi,\omega)$ is continuous if ϕ is a simple function. If we had a sequence of simple functions ϕ_n converging to f in L^2, we would like to "transfer" the continuity of ϕ to f. To do so we use the following fact: " The uniform limit of continuous functions is continuous". In other words, if f_n is continuous and $\sup_{t\in[0,T]}|f_n(t)-f(t)|\to 0$ as $n\to\infty$ then f is continuous.

To do so, let $I_t(\phi_n)=\int_0^t\phi(s)dB_s$ and consider a Cauchy sequence $\{\phi_n\}$, for which we have

$$\begin{aligned}\mathbb{P}\left[\sup_{0\leqslant t\leqslant T}|I_t(\phi_n)-I_t(\phi_m)|>\varepsilon\right]&\leqslant\frac{1}{\varepsilon^2}\mathbb{E}\left[|I_T(\phi_n)-I_T(\phi_m)|^2\right]\\&\leqslant\frac{1}{\varepsilon^2}\mathbb{E}\left[\int_0^T(\phi_n(s)-\phi_m(s))^2\,\mathrm{d}s\right]\end{aligned}$$

The last term goes to zero as $n,m\to\infty$. Hence we can find a subsequence $\{n_k\}$ so that

$$\mathbb{P}\left[\sup_{0\leqslant t\leqslant T}|I_t(\phi_{n_{k+1}})-I_t(\phi_{n_k})|>\frac{1}{k^2}\right]\leqslant\frac{1}{2^k}$$

If we set

$$A_k=\left\{\sup_{0\leqslant t\leqslant T}|I_t(\phi_{n_{k+1}})-I_t(\phi_{n_k})|>\frac{1}{k^2}\right\}$$

then $\sum_k\mathbb{P}[A_k]\leqslant\sum_k 2^{-k}<\infty$. Hence the Borel-Cantelli lemma tells us that there is an random $N(\omega)$ so that

$$n>N(\omega)\Longrightarrow\sup_{0\leqslant t\leqslant T}|I_t(\phi_{n_{k+1}})-I_t(\phi_{n_k})|\leqslant\frac{1}{k^2}$$

If we set $J_t^{(k)}=I_t(\phi_{n_k})$ then the $\{J_t^{(k)}\}$ form a Cauchy sequence in the sup norm ($|f|_{\text{sup}}=\sup_{t\in[0,T]}|f(t)|$). Since the convergence is uniform in t and each $J_t^{(k)}$ is continuous, we know that for almost every ω the limit point $\lim_{k\to\infty}J_t^{(k)}=J_t$ is also continuous in t. Finally, since by assumption we also have $J_t^{(k)}\to I_t(f)=\int_0^t f_s(\omega)\mathrm{d}B_s(\omega)$ in L^2, we have that

$$\int_0^t f_s(\omega)\mathrm{d}B_s(\omega)=J_t(\omega)\quad\text{a.s.},$$

as required. □

8. An Extension of the Itô Integral

Up until now we have only considered the Itô integral for integrands f such that $\mathbb{E}\int_0^T f^2\,\mathrm{d}s<\infty$. However it is possible to make sense of $\int_0^T f_s(\omega)\mathrm{d}B_s(\omega)$ if we only know that

$$\mathbb{P}\left[\int_0^T|f_s(\omega)|^2\,\mathrm{d}s<\infty\right]=1\ .$$

Most of the previous properties hold. In particular $\int_0^T f_s(\omega)\mathrm{d}B_s(\omega)$ is a perfectly fine random variable which is almost surely finite. However, it is not necessarily true that $\mathbb{E}[\int_0^T f_s(\omega)\mathrm{d}B_s(\omega)] = 0$. Which in turn means that $\int_0^T f_s(\omega)\mathrm{d}B_s(\omega)$ need not be a martingale. (In fact it is what is called a local martingale.) By obvious reasons, the Itô isometry property (4.12) may not hold in this case.

Example 8.1.

9. Itô Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the canonical probability space and let $\mathcal{F}_t$ be the σ-algebra generated by the Brownian motion $B_t(\omega)$.

Definition 9.1. *$X_t(\omega)$ is an Itô process if there exist stochastic processes $f(t,\omega)$ and $\sigma(t,\omega)$ such that*

i) *$f(t,\omega)$ and $\sigma(t,\omega)$ are $\mathcal{F}_t$-measurable,*
ii) *$\int_0^t |f| \,\mathrm{d}s < \infty$ and $\int_0^t |\sigma|^2 \,\mathrm{d}s < \infty$ almost surely,*
iii) *$X_0(\omega)$ is $\mathcal{F}_0$-measurable,*
iv) *With probability one the following holds*

$$X_t(\omega) = X_0(\omega) + \int_0^t f_s(\omega)\,\mathrm{d}s + \int_0^t \sigma_s(\omega)\mathrm{d}B_s(\omega) \tag{4.13}$$

The processes $f(t,\omega)$ and $\sigma(t,\omega)$ are referred to as drift *and* diffusion *coefficients of X_t.*

For brevity, one often writes (4.13) as

$$\mathrm{d}X_t(\omega) = f_t(\omega)\,\mathrm{d}t + \sigma_t(\omega)\mathrm{d}B_t(\omega)$$

But this is just notation for the integral equation above!

CHAPTER 5

Stochastic Calculus

This section introduces the fundamental tools for the computation of stochastic integrals. Indeed, similarly to what is done in classical calculus, stochastic integrals are rarely computed by applying the definition of Itô integral from the previous chapter. In the case of classical calculus, instead of applying the definition of Riemann integral one usually computes $\int f(x)\mathrm{d}x$ by applying the fundamental theorem of calculus and choosing $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$ such that

$$\int f(x)\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}F(x)\mathrm{d}x = F(x)\,. \tag{5.1}$$

Even though, as we have seen in the previous section differentiation in this framework is not possible, it is possible to obtain a similar result for Itô integrals. In the following chapter we will introduce such a formula (called the Itô formula) allowing for rapid computation of stochastic integrals.

1. Itô's Formula for Brownian motion

We first introduce the Itô formula for the Brownian motion process.

Theorem 1.1. *Let $f \in C^2(\mathbb{R})$ (the set of twice continuously differentiable functions on $\mathbb{R}$) and B_t a standard Brownian motion. Then for any $t > 0$,*

$$f(B_t) = f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s\,.$$

To prove this theorem, we first state the following partial result, proven at the end of the section.

Lemma 1.2. *Let g be a continuous function and $\Gamma^n := \{t_k^n : k = 1, \dots, N(n)\}$ be a sequence of partitions of $[0,t]$ such that $t_0 = 0$ and $t_N = t$*

$$|\Gamma^n| := \sup_i |t_{i+1}^n - t_i^n| \to 0 \qquad as \qquad n \to \infty\,. \tag{5.2}$$

Then

$$\sum_{k=0}^{N-1} g(\xi_k^n)\left(B_{t_k^n} - B_{t_{k+1}^n}\right)^2 \to \int_0^t g(B_s)\,\mathrm{d}s\,,$$

for any choice of $\xi_k^n \in (B_{t_k^n}, B_{t_{k+1}^n})$.

Proof of Theorem 1.1. Without loss of generality we can assume that f and its first two derivatives are bounded. After establishing the result for such functions we can approximate any function by such a sequence and pass to the limit to obtain the general result.

Let $\{t_k^n : k = 1, \dots, N(n)\}$ be a sequence of partitions of $[0,t]$ such that $t_0 = 0$ and $t_N = t$

$$|\Gamma^N| := \sup_i |t_{i+1}^n - t_i^n| \to 0 \qquad \text{as} \qquad n \to \infty\,.$$

Now for any level n,

$$f(B_t) - f(0) = \sum_{k=1}^{N(n)} \left(f(B_{t_k}) - f(B_{t_{k-1}})\right). \tag{5.3}$$

Taylor's Theorem implies

$$f(B_{t_k}) - f(B_{t_{k-1}}) = f'(B_{t_{k-1}})\big(B_{t_k} - B_{t_{k-1}}\big) + \frac{1}{2} f''(\xi_k)\big(B_{t_k} - B_{t_{k-1}}\big)^2 ,$$

for some $\xi_k \in [B_{t_{k-1}}, B_{t_k}]$. Returning to (5.3), we have

$$f(B_t) - f(0) = \sum_{k=1}^{N(n)} f'(B_{t_{k-1}})\big(B_{t_k} - B_{t_{k-1}}\big) + \frac{1}{2} \sum_{k=1}^{N(n)} f''(\xi_k)\big(B_{t_k} - B_{t_{k-1}}\big)^2 . \tag{5.4}$$

By the construction of the Itô integral, for the first term on the right hand side of (5.4) we have

$$\sum_{k=1}^{N(n)} f'(B_{t_{k-1}})\big(B_{t_k} - B_{t_{k-1}}\big) \longrightarrow \int_0^t f'(B_s)\, \mathrm{d}B_s$$

in L^2 as $n \to \infty$. Combining this with the result of Lemma 1.2 (proven below) with $g = f''$ for the second term on the right hand side of (5.4) we conclude the proof. □

Proof of Lemma 1.2. We now want to show that

$$A_n := \sum_{k=1}^{N(n)} g(\xi_k)\big(B_{t_k} - B_{t_{k-1}}\big)^2 \longrightarrow \int_0^t g(B_s)\, \mathrm{d}s$$

in probability as $n \to \infty$. We begin by showing that

$$C_n := \sum_{k=1}^{N(n)} g(B_{t_{k-1}})\big(B_{t_k} - B_{t_{k-1}}\big)^2 \longrightarrow \int_0^t g(B_s)\, \mathrm{d}s , \tag{5.5}$$

in probability as $n \to \infty$. First since $g(B_s)$ is continuous, we have that

$$D_n := \sum_{k=1}^{N(n)} g(B_{t_{k-1}})(t_k - t_{k-1}) \longrightarrow \int_0^t g(B_s)\, \mathrm{d}s .$$

as $n \to \infty$. Therefore, we obtain (5.5), by showing that the term $|C_n - D_n|$ converges to 0 in $L^2(\Omega)$ as this directly implies convergence in probability. To that end observe that

$$\begin{aligned} \mathbb{E}\left[(D_n - C_n)^2\right] = \mathbb{E}&\left[\sum_{k=1}^{N(n)} g(B_{t_{k-1}})^2(\Delta_k t - \Delta_k^2 B)^2\right] \\ &+ 2\mathbb{E}\left[\sum_{j<k} g(B_{t_{k-1}})g(B_{t_{j-1}})(\Delta_k t - \Delta_k^2 B)(\Delta_j t - \Delta_j^2 B)\right] \end{aligned} \tag{5.6}$$

where $\Delta_k := t_k - t_{k-1}$ and $\Delta_k^2 B := (B_{t_k} - B_{t_{k-1}})^2$. Now since

$$\mathbb{E}\left[(\Delta_k t - \Delta_k^2 B)^2\right] = (\Delta_k t)^2 - 2(\Delta_k t)^2 + 3(\Delta_k t)^2 = 2(\Delta_k t)^2$$

Considering the first term in (5.6) we have

$$\mathbb{E}\left[\sum_{k=1}^{N(n)} g(B_{t_{k-1}})^2(\Delta_k t - \Delta_k^2 B)^2\right] = 2\sum_{k=1}^{N(n)} \mathbb{E}\left[g(B_{t_{k-1}})^2(\Delta_k t)^2\right] \leqslant 2|\Gamma^N| \sum_{k=1}^{N(n)} \mathbb{E}\left[g(B_{t_{k-1}})^2(\Delta_k t)\right]$$

Since the sum converges to

$$\int_0^t \mathbb{E}\left[g(B_s)^2\right]\, \mathrm{d}s < \infty$$

as $n \to \infty$ and $|\Gamma^N| \to 0$, the product goes to zero. All that remains is the second sum in (5.6). Since

$$\mathbb{E}\left[g(B_{t_{k-1}})g(B_{t_{j-1}})(\Delta_k t - \Delta_k^2 B)(\Delta_j t - \Delta_j^2 B)\right] \\ = \mathbb{E}\left[g(B_{t_{k-1}})g(B_{t_{j-1}})(\Delta_j t - \Delta_j^2 B)\mathbb{E}\left[\Delta_k t - \Delta_k^2 B \middle| \mathcal{F}_{t_{k-1}}\right]\right]$$

and $\mathbb{E}\left[\Delta_k t - \Delta_k^2 B \middle| \mathcal{F}_{t_{k-1}}\right] = 0$, we see that the second sum is in fact zero.

All that remains is to show that A_n converges to C_n. Now

$$\begin{aligned} |C_n - A_n| &\leqslant \sum_{k=1}^{N(n)} |g(\xi_k) - g(B_{t_k})| \big(B_{t_k} - B_{t_{k-1}}\big)^2 \\ &\leqslant \Big(\sup_k |g(\xi_k) - g(B_{t_k})|\Big) \sum_{k=1}^{N(n)} \big(B_{t_k} - B_{t_{k-1}}\big)^2 . \end{aligned}$$

Since the first term goes to zero in probability as $n \to \infty$ by the continuity and boundedness of $g(B_s)$ and the second term converges to the quadratic variation of B_t (which equals t), we conclude that the product converges to zero in probability. □

1.1. A second look at Itô's Formula. Looking back at (5.4), one sees that the Itô integral term in Itô's Formula comes from the sum against the increments of Brownian motion. This term results directly from the first order Taylor expansion of f, and can be identified with the first order derivative term that we are used to see in the fundamental theorem of calculus (5.1). The second sum

$$\sum_{k=1}^{N(n)} f''(\xi_k)\big(B_{t_k} - B_{t_{k-1}}\big)^2 ,$$

which contains the squares of the increments of Brownian motion, results from the second order term in the Taylor expansion and is *absent* in the classical calculus formulation. However, since the sum

$$\sum_{k=1}^{N(n)} \big(B_{t_k} - B_{t_{k-1}}\big)^2$$

converges in probability to the quadratic variation of the Brownian motion B_t, which according to Lemma 5.3 is simply t, this term gives a nonzero contribution in the limit $n \to \infty$ and should be considered in this framework. We refer to this term as the *Itô correction term*. In light of this remark, if we let $[B]_t$ denote the quadratic variation of B_t, then one can reinterpret the Itô correction term

$$\frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s \qquad \text{as} \qquad \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}[B]_s . \tag{5.7}$$

We wish to derive a more general version of Itô's formula for a general Itô process X_t defined by

$$X_t = X_0 + \int_0^t f_s\,\mathrm{d}s + \int_0^t g_s\,\mathrm{d}B_s .$$

Beginning in the same way as before, we write the expression analogous to (5.4), namely

$$f(X_t) - f(X_0) = \sum_{k=1}^{N(n)} f'(X_{t_{k-1}})\big(X_{t_k} - X_{t_{k-1}}\big) + \frac{1}{2}\sum_{k=1}^{N(n)} f''(\xi_k)\big(X_{t_k} - X_{t_{k-1}}\big)^2 \tag{5.8}$$

for some twice continuously differentiable function f and some partition $\{t_k\}$ of $[0,t]$. It is reasonable to expect the first and second sums to converge respectively to the integrals

$$\int_0^t f'(X_s)\,\mathrm{d}X_s \qquad \text{and} \qquad \frac{1}{2}\int_0^t f''(X_s)\,d[X]_s\,. \tag{5.9}$$

The first is the Itô stochastic integral with respect to an Itô process X_t while the second is an integral with respect to the differential of the quadratic variation $[X]_t$ of the process X_t. So far this discussion has proceeded mainly by analogy to the simple Brownian motion. To make sense of the two terms in (5.9) we need to better understand the quadratic variation of an Itô process X_t and to define the concept of stochastic integral against X_t. While the former point is covered in the following section, the latter is quickly clarified by this intuitive definition:

Definition 1.3. *Given an Itô process $\{X_t\}$ with differential $\mathrm{d}X_t = f_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t$ and an adapted stochastic process $\{h_t\}$ such that*

$$\int_0^\infty |h_s f_s|\,\mathrm{d}s < \infty \qquad \text{and} \qquad \int_0^\infty (h_s\sigma_s)^2\,\mathrm{d}s < \infty \qquad a.s.$$

then we define the integral of h_t against X_t as

$$\int_0^t h_s\,\mathrm{d}X_s := \int_0^t h_s f_s\,\mathrm{d}s + \int_0^t h_s\sigma_s\,\mathrm{d}B_s\,. \tag{5.10}$$

2. Quadratic Variation and Covariation

We generalize the definition (3.7) of quadratic variation to the one of *quadratic covariation*

Definition 2.1. *Let X_t, Y_t be two adapted, stochastic processes. Their* quadratic covariation *is defined as*

$$[X,Y]_t := \lim_{N\to\infty}^{p} \sum_{j=0}^{j^N} \left(X_{t^N_{j+1}} - X_{t^N_j}\right)\left(Y_{t^N_{j+1}} - Y_{t^N_j}\right)$$

where $\lim^p$ *denotes a limit in probability and $\{t^N_j\}$ is a set partitioning the interval $[0,t]$ defined by*

$$\Gamma^N := \{\{t^N_j\} \;:\; 0 = t^N_0 < t^N_1 < \cdots < t^N_{j^N} = t\} \tag{5.11}$$

with $|\Gamma^N| := \sup_j |t^N_{j+1} - t^N_j| \to 0$ as $N \to \infty$. Furthermore, we define the quadratic variation *of X_t as*

$$[X]_t := [X,X]_t\,. \tag{5.12}$$

We can also speak about the quadratic variation on an interval different than $[0,t]$. For $0 \leqslant s < t$ we will write respectively $[X]_{s,t}$ and $[X,Y]_{s,t}$ for the quadratic variation and cross-quadratic variation on the interval $[s,t]$.
Just from the algebraic form of the pre-limiting object the quadratic variation satisfies a number of properties.

Lemma 2.2. *Assuming all of the objects are defined, then for any adapted, continuous stochastic processes X_t, Y_t,*

i) *for any constant $c \in \mathbb{R}$ we have*

$$[c\,X]_t = c^2[X]_t$$

ii) *for $0 < s < t$ we have*

$$[X]_{0,s} + [X]_{s,t} = [X]_{0,t} \tag{5.13}$$

iii) *we have that*

$$0 \leqslant [X]_{0,s} \leqslant [X]_{0,t} \tag{5.14}$$

for $t > s \geqslant 0$. In other words, the map $t \mapsto [X]_t$ is nondecreasing a.s. .

iv) *we can write*

$$[X \pm Y]_t = [X]_t + [Y]_t \pm 2[X,Y]_t\,. \tag{5.15}$$

Consequently, quadratic covariations can be written in terms of quadratic variations as

$$[X,Y]_t = \frac{1}{2}\Big([X+Y]_t - [X]_t - [Y]_t\Big) = \frac{1}{4}\Big([X+Y]_t - [X-Y]_t\Big)\,. \tag{5.16}$$

Proof. Parts i) and ii) are a direct consequence of Def. 2.1, while part iii) results from ii) the fact that $[X,Y]_t$ is defined as a sum of squares and is therefore nonnegative: $[X]_{0,t} = [X]_{0,s} + [X]_{s,t} \geqslant [X]_{0,s}$. Part iv) is obtained by noticing that

$$\begin{aligned}
[X \pm Y]_t &= \overset{p}{\lim_{N\to\infty}} \sum_{j=0}^{j^N} \Big((X_{t^N_{j+1}} - X_{t^N_j}) \pm (Y_{t^N_{j+1}} - Y_{t^N_{j+1}})\Big)^2 \\
&= \overset{p}{\lim_{N\to\infty}} \sum_{j=0}^{j^N} \left(\left(X_{t^N_{j+1}} - X_{t^N_j}\right)^2 \pm 2\left(X_{t^N_{j+1}} - X_{t^N_j}\right)\left(Y_{t^N_{j+1}} - Y_{t^N_j}\right) + \left(Y_{t^N_{j+1}} - Y_{t^N_j}\right)^2\right) \\
&= [X]_t \pm 2[X,Y]_t + [Y]_t\,,
\end{aligned}$$

while (5.16) is obtained by rearranging the terms of the above result (for the first inequality) and by using it to compute $[X+Y]_t + [X-Y]_t$ (for the second). ☐

In the following sections we will see that the quadratic variation of Itô integrals acquires a particularly simple form. We will show this by first considering quadratic variations of Itô integrals and then extend this result to Itô processes.

2.1. Quadratic Variation of an Itô Integral.

Lemma 2.3. *Let σ_t be a process adapted to the filtration $\{\mathcal{F}^B_t\}$ and such that $\int_0^\infty \sigma_s^2\,\mathrm{d}s < \infty$ a.s.. Then defining $M_t := I_t(\sigma) = \int_0^t \sigma_s\,\mathrm{d}B_s$ we have that*

$$[M]_t = \int_0^t \sigma_s^2\,\mathrm{d}s \tag{5.17}$$

or in differential notation $\mathrm{d}[M]_t = \sigma_t^2\,\mathrm{d}t$.

Proof of Lemma 2.3. It is enough to prove (5.17) when σ_s is an elementary stochastic process in $\mathcal{S}_2$. The general case can then be handled by approximation as in the proof of the Itô isometry. Hence, we assume that

$$\sigma_t = \sum_{j=1}^{K} \alpha_{j-1}\mathbf{1}_{[t_{j-1},t_j)}(t) \tag{5.18}$$

where the α_k satisfy the properties required by $\mathcal{S}_2$ and K is some integer. Without loss of generality we can assume that t is the right endpoint of our interval so that the partition takes the form

$$0 = t_0 < t_1 < \cdots < t_K = t$$

Now observe that if $[s,r] \subset [t_{j-1},t_j]$ then

$$\int_s^r \sigma_\tau\,\mathrm{d}B_\tau = \alpha_{j-1}\big(B_s - B_r\big)$$

Hence $\{s_\ell^{(n)}\}$ is a sequence of partitions of the interval $[t_{j-1}, t_j]$ so that

$$t_{j-1} = s_0^{(n)} < s_1^{(n)} < \cdots < s_{N(n)}^{(n)} = t_j$$

and $|\Gamma^N| = \sup_\ell |s_{\ell-1}^{(n)} - s_\ell^{(n)}| \to 0$ as $n \to \infty$. Then the quadratic variation of M_t on the interval $[t_{j-1}, t_j]$ is the limit $n \to \infty$ of

$$\sum_{\ell=1}^{N(n)} (M_{s_{\ell-1}} - M_{s_\ell})^2 = \alpha_{j-1}^2 \sum_{\ell=1}^{N(n)} (B_{s_{\ell-1}} - B_{s_\ell})^2 .$$

Since the summation on the right hand side limits to the quadratic variation of the Brownian motion B on the interval $[t_{j-1}, t_j]$ which we know to be $t_j - t_{j-1}$ we conclude that

$$[M]_{t_{j-1},t_j} = \alpha_{j-1}^2 (t_j - t_{j-1}) .$$

Since the quadratic variation on disjoint intervals adds, we have that

$$[M]_t = \sum_{j=1}^K [M]_{t_{j-1},t_j} = \sum_{j=1}^K \alpha_{j-1}^2 (t_j - t_{j-1}) = \int_0^t \sigma_s^2 \, \mathrm{d}s$$

where the last equality follows from the fact that σ_s takes the form (5.18). As mentioned at the start, the general form follows from this calculation by approximation by functions in $\mathcal{S}_2$. $\square$

REMARK 2.4. *The proof of the above result in Klebaner (Theorem 4.14 on pp. 106) has a subtle issue . When bounding*

$$2\sum_{i=0}^{n-1} g^2(B_{t_i^n})(t_{i+1}^n - t_i^n)^2 \leqslant 2\delta_n \mathbb{E}\left[\sum_{i=0}^{n-1} g^2(B_{t_i^n})(t_{i+1}^n - t_i^n)\right],$$

Klebaner asserts that as $n \to \infty$, $\delta_n = |\Gamma_n| \to 0$, $\sum_{i=0}^{n-1} \mathbb{E}[g^2(B_{t_i^2})](t_{i+1}^n - t_i^n)$ would stay finite, and thus their product would go to 0. However, the finiteness of $\sum_{i=0}^{n-1} \mathbb{E}[g^2(B_{t_i^2})](t_{i+1}^n - t_i^n)$ is unjustified. In fact, if it were finite, it must converge to $\int_0^t \mathbb{E}[g^2(B_s)]ds$ (Riemann sum). However, this integral might be infinity for certain choice of g, for example, $g(x) = e^{x^2}$ (see Example 4.5 on pp. 99 of [Klebaner]). The proof here uses the same computation of second moment but only for "nice" functions (i.e., those with compact support). The convergence in probability (note: this is weaker than convergence in L^2) for general continuous functions is established using approximation. The stopping rules for Itô integral are needed here, but we defer it to the later part of the course.

We now consider the quadratic covariation of two Itô integrals with respect to independent Brownian motions.

LEMMA 2.5. *Let B_t, W_t two independent Brownian motions, and f_s, g_s two stochastic processes, all adapted to the underlying filtration $\mathcal{F}_t$ and such that $\int_0^\infty f_s^2 \, \mathrm{d}s, \int_0^\infty g_s^2 \, \mathrm{d}s < \infty$ almost surely. We define*

$$M_t := \int_0^t f_s \, \mathrm{d}B_s \qquad \text{and} \qquad N_t := \int_0^t g_s \, \mathrm{d}W_s .$$

Then, for all $t \geqslant 0$ one has

$$[N, M]_t = 0 . \tag{5.19}$$

PROOF OF (5.19). Again without lost of generality it is enough to prove the result for σ_t and g_s in $\mathcal{S}_2$. We can further assume that both functions are defined with respect to the same partition

$$0 = t_0 < t_1 < \cdots < t_K = t$$

Since as observed in (5.13), the quadratic variation on disjoint intervals adds, we need only show that $[N,M]_{t_{j-1},t_j}=0$ on any of the partition intervals $[t_{j-1},t_j]$.

Fixing such an interval $[t_{j-1},t_j]$, we see that $[N,M]_{t_{j-1},t_j}=\sigma_{t_{j-1}}g_{t_{j-1}}[W,B]_{t_{j-1},t_j}$. The easiest way to see this is to use the "polarization" equality (5.16)

$$2[W,B]_{t_{j-1},t_j}=[\tfrac{W+B}{\sqrt{2}},\tfrac{W+B}{\sqrt{2}}]_{t_{j-1},t_j}-[\tfrac{W-B}{\sqrt{2}},\tfrac{W-B}{\sqrt{2}}]_{t_{j-1},t_j}=(t_j-t_{j-1})-(t_j-t_{j-1})=0$$

since $\frac{W+B}{\sqrt{2}}$ and $\frac{W-B}{\sqrt{2}}$ are standard Brownian motions and hence have quadratic variation the length of the time interval. $\square$

REMARK 2.6. *One can also prove the above result more directly by following the same argument as in Theorem 5.3 of Chapter 3. The key calculation is to show that the expected value of the approximating quadratic variation is 0 and not the length of the time interval as in the proof of Theorem* (5.3) *of Chapter 3. For any partitions* $\{s_\ell\}$ *of* $[t_{j-1},t_j]$ *one has to zero we have*

$$\mathbb{E}\sum_\ell(B_{s_\ell}-B_{s_\ell-1})(B_{s_\ell}-B_{s_\ell-1})=\sum_\ell\mathbb{E}(B_{s_\ell}-B_{s_\ell-1})\mathbb{E}(B_{s_\ell}-B_{s_\ell-1})=0$$

2.2. Quadratic Variation of an Itô Process. In this section, using the results presented above, we finally obtain a simple expression for the quadratic variation of an Itô process.

LEMMA 2.7. *If X_t is an Itô process with differential $\mathrm{d}X_t=\mu_t\,\mathrm{d}t+\sigma_t\,\mathrm{d}B_t$, then*

$$[X]_t=[I(\sigma^2)]_t=\int_0^t\sigma_s^2\,\mathrm{d}s\,, \tag{5.20}$$

or equivalently $\mathrm{d}[X]_t=\sigma_t^2\,\mathrm{d}t$.

By comparing this result with (5.17) we notice that the only contribution to the quadratic variation process comes from the Itô integral. The following result will be useful in the proof of (5.20).

LEMMA 2.8. *Let X_t and Y_t be adapted stochastic processes, such that X_t is continuous a.s. and Y_t has trajectories with finite first variation ($V_1[Y](t)<\infty$) then $[X,Y]_t=0$ a.s..*

Before we give the proof of Lemma 2.8 we observe that it immediately yields (5.20).

PROOF OF LEMMA 2.7. Defining $F_t=\int_0^t\mu_s\,\mathrm{d}s$ and $M_t=\int_0^t\sigma_s\mathrm{d}B_s$, observe that $X_t=F_t+M_t$ and that F_t is continuous and of finite first variation almost surely. Hence $[F]_t=0$. Since M_t is continuous a.s., we have that $[M,F]_t=0$ almost surely. Hence

$$[X]_t=[F]_t+2[F,M]_t+[M]_t=[M]_t=\int_0^t\sigma_s^2\,\mathrm{d}s\,.$$

$\square$

PROOF OF LEMMA 2.8 . Let $\Gamma^N:=\{t_i^N:i=0,\dots,i_N\}$ be a sequence of partitions of $[0,t]$ such that $|\Gamma^N|=\sup_i|t_{i+1}^N-t_i^N|\to0$ as $N\to\infty$. Now

$$\Big|\sum_{i=1}^{i^N}(X_{t_i}-X_{t_{i-1}})(Y_{t_i}-Y_{t_{i-1}})\Big|\leqslant\Big(\sup_i|X_{t_i}-X_{t_{i-1}}|\Big)\sum_{i=1}^{i^N}|Y_{t_i}-Y_{t_{i-1}}|$$

The summation on the right hand side is bounded from above by the first variation of Y_t which by assumption is finite a.s. On the other hand, as $n\to\infty$ the supremum goes to zero since $|\Gamma^N|\to0$ and X_t is a.s. continuous. $\square$

REMARK 2.9. *Similarly to the formal considerations in Section 1.1, we may think of the differential of the quadratic variation process* $\mathrm{d}[X]_t$ *as the limit of the difference term* $(X_{t^n_{k+1}} - X_{t^n_k})^2$*, which in turn is the square of the differential of Brownian motion* $(\mathrm{d}X_t)^2$*. Therefore,* formally speaking*, we can obtain the result of the previous lemma by writing*

$$\begin{aligned} \mathrm{d}[X]_t &= (\mathrm{d}X_t)^2 = (\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t)^2 \\ &= \mu_t^2(\mathrm{d}t)^2 + 2\mu_t\sigma_t(\mathrm{d}t)(\mathrm{d}B_t) + \sigma_t^2(\mathrm{d}B_t)^2 \\ &= \sigma_t^2\,\mathrm{d}t\,. \end{aligned}$$

where we have applied $(\mathrm{d}t)^2 = (\mathrm{d}t)(\mathrm{d}B_t) = 0$ *(cfr. Lemma 2.8) and* $(\mathrm{d}B_t)^2 = \mathrm{d}t$ *(cfr. Lemma 2.3). These* formal *multiplication rules are summarized in the following table: By the same formal*

$\times$	$\mathrm{d}t$	$\mathrm{d}B_t$
$\mathrm{d}t$	*0*	*0*
$\mathrm{d}B_t$	*0*	$\mathrm{d}t$

arguments, such rules apply to the computation of the quadratic covariation of two Itô processes $Y_t = \mu'_t\,\mathrm{d}t + \sigma'_t\mathrm{d}B_t$*:*

$$\begin{aligned} \mathrm{d}[X,Y]_t &= (\mathrm{d}X_t)(\mathrm{d}Y_t) = (\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t)(\mu'_t\,\mathrm{d}t + \sigma'_t\,\mathrm{d}B_t) \\ &= \mu_t\mu'_t(\mathrm{d}t)^2 + \mu_t\sigma'_t(\mathrm{d}t)(\mathrm{d}B_t) + \mu'_t\sigma_t(\mathrm{d}t)(\mathrm{d}B_t) + \sigma_t\sigma'_t(\mathrm{d}B_t)^2 \\ &= \sigma_t\sigma'_t\,\mathrm{d}t\,. \end{aligned} \tag{5.21}$$

This result can be verified by going through the steps of the proof of the above lemmas.

3. Itô's Formula for an Itô Process

Using the lessons learned in the previous sections, we now proceed to compute the infinitesimal of (5.8) as

$$\mathrm{d}f(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(\mathrm{d}X_t)^2\,. \tag{5.22}$$

Of course this is just notation for the integral equation

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)(\mathrm{d}X_s)^2\,.$$

The first integral is simply the integral against an Itô process as we have already discussed. In light of Remark 2.9, we should interpret $\mathrm{d}[X_t] = (\mathrm{d}X_t)^2 = (\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t)^2 = \sigma_t^2\mathrm{d}t$. Hence

$$\frac{1}{2}\int_0^t f''(X_s)(\mathrm{d}X_s)^2 = \frac{1}{2}\int_0^t f''(X_s)\sigma_s^2\,\mathrm{d}s$$

This formal calculation (which is correct) leads us to suggest the following general Itô formula.

THEOREM 3.1. *Let* X_t *be the Itô process given by*

$$\mathrm{d}X_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t$$

If f *is a* C^2 *function then*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}[X]_s \\ &= f(X_0) + \int_0^t f'(X_s)\mu_s\,\mathrm{d}s + \int_0^t f'(X_s)\sigma_s\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(X_s)\sigma_s^2\,\mathrm{d}s \end{aligned}$$

Proof of Theorem 3.1. Without loss of generality, we can assume that both μ_t and σ_t are adapted elementary stochastic processes satisfying the assumptions required by an Itô process. Furthermore, by inserting partition points if needed, we can assume that they are both defined on the same partition

$$0 = t_0 < t_1 < \cdots < t_N = t\,.$$

Since

$$f(X_t) - f(X_0) = \sum_{\ell=1}^{N} f(X_{t_\ell}) - f(X_{t_{\ell-1}})$$

we need only prove Itô's formula for $f(X_{t_\ell}) - f(X_{t_{\ell-1}})$. Now since X_t is constant for $t \in [t_{\ell-1}, t_\ell)$ we can take $\xi = X_{t_{\ell-1}}$ and for $[r,s] \subset [t_{\ell-1}, t_\ell]$ we have

$$X_s - X_r = \int_r^s \mu_\tau d\tau + \int_r^s \sigma_\tau \,\mathrm{d}B_\tau = \mu_{t_{\ell-1}}(s-r) + \sigma_{t_{\ell-1}}(B_s - B_r)\,.$$

Let $\{s_i^{(n)} : i = 0, \ldots, K(n)\}$ be a sequence of partitions of $[t_{\ell-1}, t_\ell]$ such that $|\Gamma^N| = \sup_i |s_i^{(n)} - s_{i-1}^{(n)}| \to 0$ as $n \to \infty$. Now using Taylor's theorem we have

$$f(X_{s_j}) - f(X_{s_{j-1}}) = f'(X_{s_{j-1}})(X_{s_j} - X_{s_{j-1}}) + \frac{1}{2} f''(\xi_j)(X_{s_j} - X_{s_{j-1}})^2\,.$$

for some $\xi_j \in (X_{s_{j-1}}, X_{s_j})$. Hence we have

$$\begin{aligned}
f(X_{t_\ell}) - f(X_{t_{\ell-1}}) &= \sum_{j=1}^{K(n)} f(X_{s_j}) - f(X_{s_{j-1}}) \\
&= \sum_{j=1}^{K(n)} f'(X_{s_{j-1}})(X_{s_j} - X_{s_{j-1}}) + \frac{1}{2} \sum_{j=1}^{K(n)} f''(\xi_j)(X_{s_j} - X_{s_{j-1}})^2 = (I) + \frac{1}{2}(II)
\end{aligned}$$

Since

$$(X_{s_j} - X_{s_{j-1}})^2 = \mu_{t_{\ell-1}}^2 (s_j - s_{j-1})^2 + 2\mu_{t_{\ell-1}}\sigma_{t_{\ell-1}}(s_j - s_{j-1})(B_{s_j} - B_{s_{j-1}}) + \sigma_{t_{\ell-1}}^2 (B_{s_j} - B_{s_{j-1}})^2$$

we have

$$(I) = \mu_{t_{\ell-1}} \sum_{j=1}^{K(n)} f'(X_{s_{j-1}})(s-r) + \sigma_{t_{\ell-1}} \sum_{j=1}^{K(n)} f'(X_{s_{j-1}})(B_s - B_r) = (Ia) + (Ib)$$

and

$$\begin{aligned}
(II) =& \mu_{t_{\ell-1}}^2 \sum_{j=1}^{K(n)} f''(\xi_j)(s_j - s_{j-1})^2 + 2\mu_{t_{\ell-1}}\sigma_{t_{\ell-1}} \sum_{j=1}^{K(n)} f''(\xi_j)(s_j - s_{j-1})(B_{s_j} - B_{s_{j-1}}) \\
&+ \sigma_{t_{\ell-1}}^2 \sum_{j=1}^{K(n)} f''(\xi_j)(B_{s_j} - B_{s_{j-1}})^2 = (IIa) + (IIb) + (IIc)
\end{aligned}$$

As $n \to \infty$, it is clear that

$$(Ia) \longrightarrow \int_{t_{\ell-1}}^{t_\ell} f'(X_s) b_s \,\mathrm{d}s, \qquad \text{and} \qquad (Ib) \longrightarrow \int_{t_{\ell-1}}^{t_\ell} f'(X_s)\sigma_s dB_s\,.$$

Using the same arguments as in Theorem 1.1 we see that

$$(IIc) \longrightarrow \sigma_{t_{\ell-1}}^2 \int_{t_{\ell-1}}^{t_\ell} f''(X_s)\,\mathrm{d}s = \int_{t_{\ell-1}}^{t_\ell} \sigma_s^2 f''(X_s)\,\mathrm{d}s\,.$$

All that remains is to show that (IIa) and (IIb) converge to zero as $n \to \infty$. Observe that

$$|(IIa)| \leqslant \mu_{t_{\ell-1}}^2 |\Gamma^N| \sum_{j=1}^{K(n)} f''(\xi_j)(s_j - s_{j-1}) \quad \text{and} \quad |(IIb)| \leqslant 2\mu_{t_{\ell-1}}\sigma_{t_{\ell-1}} |\Gamma^N| \sum_{j=1}^{K(n)} f''(\xi_j)(B_{s_j} - B_{s_{j-1}})\,.$$

Since the two sums converge to $\int f''(X_s)\,\mathrm{d}s$ and $\int f''(X_s) dB_s$ respectively the fact that $|\Gamma^N| \to 0$ implies that (IIa) and (IIb) converge to zero as $n \to \infty$. Putting all of these results together produces the quoted result. □

Remark 3.2. *Notice that the "multiplication table" given in Remark 2.9 is reflected in the details of the proof of Theorem 3.1. Each of the terms in* $(\mathrm{d}X_t)^2$ *correspond to one of the terms labeled* (II) *which came from* $(X(s_j) - X(s_{j-1}))^2$ *in the Taylor's theorem expansion. The term* (IIa) *which corresponds to* $(\mathrm{d}t)^2$ *limits to zero as the multiplication table indicates. The term* (IIb) *which corresponds to* $(\mathrm{d}t)(\mathrm{d}B_t)$ *also tends to zero again as the table indicates. Lastly,* (IIc) *which corresponds to* $(\mathrm{d}B_t)^2$ *limits to an integral against* $(\mathrm{d}t)$ *as indicated in the table.*

Example 3.3. *Consider the stochastic process with differential*

$$\mathrm{d}X_t = \frac{1}{2} X_t\,\mathrm{d}B_t + X_t\,\mathrm{d}B_t\,.$$

The above process is an example of a geometric Brownian motion, *a process widely used in finance to model the price of a stock. We apply Itô formula to the function* $f(x) := \log x$. *Using that* $\partial_x f(x) = x^{-1}$ *and* $\partial_{xx}^2 f(x) = -x^{-2}$ *we obtain*

$$\mathrm{d}\log X_t = \frac{1}{X_t}\,\mathrm{d}X_t - \frac{1}{2}\frac{1}{X_t^2}\,\mathrm{d}[X]_t = \frac{1}{X_t}\left(\frac{1}{2}X_t\,\mathrm{d}B_t + X_t\,\mathrm{d}B_t\right) - \frac{1}{2}\frac{1}{X_t^2}\left(X_t^2\,\mathrm{d}t\right) = \mathrm{d}B_t\,.$$

In the integral form the above can be written as $\log X_t = \log X_0 + B_t$ *and therefore* $X_t = X_0 e^{B_t}$.

4. Full Multidimensional Version of Itô Formula

We now give the full multidimensional version of Itô's formula. We will include the possibility that the function depends on time and that there is more than one Brownian motion. We begin with the definition of a multidimensional Itô process.

Definition 4.1. *A stochastic process* $X_t = (X_1(t), \dots, X_d(t)) \in \mathbb{R}^d$ *is an Itô process if each of the coordinate process* $X_i(t)$ *is a one-dimensional Itô process.*

Let $\mu_i(t)$ and $\sigma_{ij}(t)$ be adapted stochastic processes and $\{B_j(t)\}_{j=1}^m$ be m mutually independent, standard Brownian motions such that for $i = 1, \dots, d$ we can write each component of the d-dimensional Itô process as

$$\mathrm{d}X_i(t) = \mu_i(t)\,\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t)\,\mathrm{d}B_j(t)\,. \tag{5.23}$$

If we collect the Brownian motions into one m-dimensional Brownian motion $B_t = (B_1(t), \dots, B_m(t))$ and define the $\mathbb{R}^d$-valued process $\mu_t = (\mu_1(t), \dots, \mu_d(t))$ and the matrix valued process σ_t whose matrix elements are the $\sigma_{ij}(t)$ then we can write

$$\mathrm{d}X_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t\,. \tag{5.24}$$

While this is nice and compact, it is perhaps more suggestive to define the $\mathbb{R}^d$-valued processes $\sigma^{(j)} = (\sigma_{1,j}, \dots, \sigma_{n,j})$ for $j = 1, \dots, m$ and write

$$\mathrm{d}X_t = \mu_t \,\mathrm{d}t + \sum_{j=1}^{d} \sigma_t^{(j)} \,\mathrm{d}B_j(t)\,. \tag{5.25}$$

This emphasizes that the process X_t at each moment of time is pushed in the direction which μ_t points and given m random kicks, in the directions the $\sigma_t^{(j)}$ point, whose magnitude and sign are dictated by the Brownian motions $B_j(t)$.

We now want to derive the process which describes evolution of $F(X_t)$ where $F : \mathbb{R}^d \to \mathbb{R}$. In other words, the multidimensional Itô formula.

We begin by developing some intuition. Recall Lemma 2.5 stating that the cross-quadratic variation of independent Brownian motions is zero. Hence if B_t and W_t are independent standard Brownian motions then the multiplication table for $(\mathrm{d}X_t)^2$ and $(\mathrm{d}Y_t)(\mathrm{d}X_t)$ if X_t and Y_t are two Itô processes is given in the following table.

$\times$	$\mathrm{d}t$	$\mathrm{d}B_t$	$\mathrm{d}W_t$
$\mathrm{d}t$	0	0	0
$\mathrm{d}B_t$	0	$\mathrm{d}t$	0
$\mathrm{d}W_t$	0	0	$\mathrm{d}t$

Table 1. Formal multiplication rules for differentials of two independent Brownian motions

Theorem 4.2. *Let $F : \mathbb{R}^d \to \mathbb{R}$ be a function such that $F(x) \in C^2$ in $x \in \mathbb{R}^d$. If X_t is as above then*

$$\mathrm{d}F(X_t) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(X_t)\mathrm{d}X_i(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial F}{\partial x_i \partial x_k}(X_t)\,\mathrm{d}[X_i, X_k]_t \tag{5.26}$$

Furthermore one has

$$\sum_{i=1}^{d}\sum_{k=1}^{n} \frac{\partial F}{\partial x_i \partial x_k}(X_t) d[X_i, X_k](t) = \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial F}{\partial x_i \partial x_k}(X_t) a_{ik}(t)\,\mathrm{d}t \tag{5.27}$$

where

$$a_{ik}(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\sigma_{kj}(t)$$

The matrix $a(t)$ can be written compactly as $\sigma(t)\sigma(t)^T$. The matrix a is often called the diffusion matrix.

We will only sketch the proof of this version of Itô formula since it follows the same logic of the others already proven. Proofs can be found in many places including [**14, 7, 3**].

Sketch of proof. Similarly to the proof of Theorem 3.1 we introduce the family of partitions Γ_N of the interval $[0, t]$ as in (8.8) with $\lim_{n\to\infty} |\Gamma_N| = 0$ and expand in Taylor the function f in each of these intervals:

$$F(X_t) - F(X_0) = \sum_{\ell=1}^{N}\Big\{ \sum_{i=1}^{d} \frac{\partial}{\partial x_i} F(X_{s_{\ell-1}})(X_i\,(s_\ell) - X_i\,(s_{j-1}))$$

$$+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}F(\xi_\ell)(X_i\,(s_\ell)-X_i\,(s_{\ell-1}))(X_j\,(s_\ell)-X_j\,(s_{\ell-1}))\Big\}$$

$$=\sum_{\ell=1}^{N}\{(I)_\ell+\frac{1}{2}(II)_\ell\}\,,$$

for $\xi_\ell \in \times_{i=1}^{d}[X_i(s_\ell), X_i(s_{\ell+1})]$. For the first order term, it is straightforward to generalize the proof of Theorem 3.1 to obtain that

$$\lim_{N\to\infty}^{p}\sum_{\ell=1}^{N}(I)_\ell=\int_0^t\sum_{i=1}^{d}\frac{\partial}{\partial x_i}F(X_s)\,\mathrm{d}X_i(s)\,.$$

We formally recover the expression of the second order term by combining (5.21) with the rules of Table 1:

$$\begin{aligned}\lim_{N\to\infty}^{p}\sum_{\ell=1}^{N}(II)_\ell &= \int_0^t\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i\partial x_j}F(X_s)(\mathrm{d}X_i(s))(\mathrm{d}X_j(s))\\ &= \int_0^t\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i\partial x_j}F(X_s)\sum_{k,l=1}^{m}(\sigma_{ik}(t)\mathrm{d}B_k(s))(\sigma_{jl}(t)\mathrm{d}B_l(s))\\ &= \int_0^t\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i\partial x_j}F(X_s)\sum_{k=1}^{m}\sigma_{ik}(t)\sigma_{jk}(t)\,\mathrm{d}t\,,\end{aligned}$$

where in the second equality we have used that $(\mathrm{d}t)(\mathrm{d}B_j(t)) = 0$ and in the third that $(\mathrm{d}B_i(t))(\mathrm{d}B_j(t)) = 0$ for $i \neq j$. Note that one should check that when taking the limit in the first equality $F(\xi_\ell)$ can be replaced by $F(X_{s_\ell})$. This can be done by reproducing the proof of Lemma 1.2. ☐

Remark 4.3. *The fact that* (5.27) *holds requires that B_i and B_j are independent if $i \neq j$. However,* (5.26) *holds even when they are not independent.*

Theorem 4.2 is for a scalar valued function F. However by applying the result to each coordinate function $F_i : \mathbb{R}^d \to \mathbb{R}$ of the function $F : \mathbb{R}^d \to \mathbb{R}^p$ with $F = (F_1, \ldots, F_p)$ we obtain the full multidimensional version. Instead of writing it again in coordinates, we take the opportunity to write a version aligned with the perspective and notation of (5.26). Recalling that the directional derivative of F in the direction $\nu \in \mathbb{R}^d$ at the point $x \in \mathbb{R}^d$ is

$$DF(x)[\nu]=\lim_{\varepsilon\to 0}\frac{F(x+\varepsilon\nu)-F(x)}{\varepsilon}=(\nabla F\cdot\nu)(x)=\sum_{k=1}^{p}\sum_{i=1}^{d}\frac{\partial F_k}{\partial x_i}\nu_i e_k$$

where e_k is the k-th unit vector of $\mathbb{R}^p$. Similarly, the second directional derivative at the point $x \in \mathbb{R}^d$ in the directions $\nu, \eta \in \mathbb{R}^d$ is given by

$$D^2F(x)[\nu,\eta]=\lim_{\varepsilon\to 0}\frac{DF(x+\varepsilon\eta)[\nu]-DF(x)[\nu]}{\varepsilon}=\sum_{k=1}^{p}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2 F_k}{\partial x_i x_j}\nu_i\eta_j e_k$$

Then in the notation of (5.26), Theorem 4.2 can be rewritten as

$$\begin{aligned}\mathrm{d}F(X(t)) = DF(X(t))[f(t)]\,\mathrm{d}t+\sum_{i=1}^{d}DF(X(t))[\sigma^{(i)}(t)]\,\mathrm{d}B_i(t)\\ +\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}D^2F(X(t))[\sigma^{(i)}(t),\sigma^{(j)}(t)]d[B_i,B_j](t)\end{aligned}$$

If B_i and B_j are assumed to be independent if $i \neq j$, then

$$\mathrm{d}F(X(t)) = DF(X(t))[f(t)]\,\mathrm{d}t + \sum_{i=1}^{d} DF(X(t))[\sigma^{(i)}(t)]\,\mathrm{d}B_i(t) + \frac{1}{2}\sum_{i=1}^{d} D^2F(X(t))[\sigma^{(i)}(t), \sigma^{(i)}(t)]\,\mathrm{d}t\,.$$

We now consider special cases of Theorem 3.1 that will be helpful in practice. The first describes evolution of a function $F(x,t)$ that depends explicitly on time:

Corollary 4.4. *Let $F : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ such that $F(x,t)$ is C^2 in $x \in \mathbb{R}^d$ and C^1 in $t \in [0,\infty)$. Furthermore, let X_t be a d-dimensional Itô process as in (5.23). Then*

$$\mathrm{d}F(X_t,t) = \frac{\partial F}{\partial t}(X_t,t)\,\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(X_t,t)\mathrm{d}X_i(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial F}{\partial x_i \partial x_k}(X_t,t)\,\mathrm{d}[X_i,X_k]_t\,. \tag{5.28}$$

where a is defined as in (5.27).

Proof. In this proof we will make use of the "multiplication table" at the beginning of this section. Consider the $d+1$-dimensional process $\bar{X}_t := (X_1(t),\ldots,X_d(t),Y_t)$ for Y_t given by $\mathrm{d}Y_t = \mathrm{d}t + 0\,\mathrm{d}B_t$. Then by applying Itô's Formula Theorem 5.26 and the fact that Y_t is of finite variation (so $\mathrm{d}[Z,Y]_t = 0$ for continuous Z_t) we obtain

$$\begin{aligned}
\mathrm{d}F(X_t) &= \sum_{i=1}^{d} \partial_i F(X_t,t)\mathrm{d}X_i(t) + \partial_t F(X_t,t)(\mathrm{d}t + 0\,\mathrm{d}B_t) \\
&\quad + \frac{1}{2}\left(\sum_{i,j=1}^{d} \partial_{ij}^2 F(X_t,t)\mathrm{d}[X_i,X_j]_t + 2\sum_{i=1}^{d} \partial_i\partial_t F(X_t,t)\mathrm{d}[X_i,Y_t] + \partial_t^2 F(X_t,t)\mathrm{d}[Y_t,Y_t]\right) \\
&= \sum_{i=1}^{d} \partial_i F(X_t,t)\mathrm{d}X_i(t) + \partial_t F(X_t,t)\mathrm{d}t + \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 F(X_t,t)\mathrm{d}[X_i,X_j]_t\,.
\end{aligned}$$

Note that the existence of the second derivative in t is not needed in the above formula and can therefore be dropped. □

Corollary 4.5. *Let X_t, Y_t be two Itô processes. Then*

$$\mathrm{d}(X_tY_t) = Y_t\mathrm{d}X_t + X_t\mathrm{d}Y_t + \mathrm{d}[X,Y]_t\,. \tag{5.29}$$

This result is known as stochastic integration by parts formula.

Proof. Let $F : \mathbb{R}^2 \to \mathbb{R}$ with $F(x,y) = x \cdot y$, then since

$$\partial_x F(x,y) = y\,, \quad \partial_y F(x,y) = x\,, \quad \partial_{xx}^2 F(x,y) = \partial_{yy}^2 F(x,y) = 0\,, \quad \partial_{xy}^2 = 1\,,$$

by Itô's Formula Theorem 5.26 we have

$$\mathrm{d}(X_tY_t) = \mathrm{d}F(X_t,Y_t) = Y_t\mathrm{d}X_t + X_t\mathrm{d}Y_t + 1\mathrm{d}[X,Y]_t\,. \tag{5.30}$$

□

Example 4.6. *We compute the stochastic integral*

$$\int_0^t s\mathrm{d}B_s\,.$$

Applying the integration by parts formula (5.30) for $\mathrm{d}X_t = \mathrm{d}B_t$, $\mathrm{d}Y_t = \mathrm{d}t$ we obtain

$$\mathrm{d}(tB_t) = t\mathrm{d}B_t + B_t\mathrm{d}t + \mathrm{d}[B,t]_t\,.$$

Since Y_t is of finite variation we have $\mathrm{d}[B, Y]_t = 0$, and by integrating and rearranging the terms we obtain

$$\int_0^t s \mathrm{d}B_s = \int_0^t \mathrm{d}(sB_s) - \int_0^t B_s \mathrm{d}s = tB_t - \int_0^t B_s \mathrm{d}s \,.$$

EXAMPLE 4.7. *Assume that $f(x,t) \in C^{2,1}(\mathbb{R}, \mathbb{R}_+)$ satisfies the* PDE

$$\frac{\partial}{\partial t} f(x,t) + \frac{\partial^2}{\partial x^2} f(x,t) = 0\,,$$

and $\mathbb{E}\left[f(B_t,t)^2\right] < \infty$, then have that

$$\begin{aligned}\mathrm{d}f(B_t,t) &= \partial_t f(B_t,t)\,\mathrm{d}t + \partial_x f(B_t,t)\,\mathrm{d}t + \frac{1}{2}\partial^2_{xx} f(B_t,t)\,\mathrm{d}[B]_t \\ &= \left(\partial_t + \frac{1}{2}\partial^2_{xx}\right) f(B_t,t)\,\mathrm{d}t + \partial_x f(B_t,t)\,\mathrm{d}B_t = \partial_x f(B_t,t)\mathrm{d}B_t\,.\end{aligned}$$

Therefore $f(B_t,t) = f(0,0) + \int_0^t \partial_x f(B_s,s)\,\mathrm{d}B_s$ is a martingale.

5. Collection of the Formal Rules for Itô's Formula and Quadratic Variation

We now recall some of the formal calculations, bringing them all together in one place. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_t$. We assume that B_t and B_t are independent standard Brownian motions adapted to the filtration $\mathcal{F}_t$.

For any $\rho \in [0,1]$, $Z_t = \rho B_t + \sqrt{1-\rho^2} B_t$ is again a standard Brownian motion. Furthermore

$$\begin{aligned}[Z]_t = [Z,Z]_t &= \rho^2 [B,B]_t + 2\rho\sqrt{1-\rho^2} = [B,B]_t + (1-\rho^2)[B,B]_t \\ &= \rho^2 t + 0 + (1-\rho^2)t = t\end{aligned}$$

Or in the formal differential notation, $d[Z]_t = \mathrm{d}t$. This result can be understood by using the formal multiplication table for differentials which formally states:

$$\begin{aligned}d[Z]_t = (dZ_t)^2 &= (\rho dB_t + \sqrt{1-\rho^2}\,\mathrm{d}B_t)^2 = \rho^2 (dB_t)^2 + 2\rho\sqrt{1-\rho^2}\,\mathrm{d}B_t dB_t + (1-\rho^2)(\,\mathrm{d}B_t)^2 \\ &= \rho^2\,\mathrm{d}t + 0 + (1-\rho^2)\,\mathrm{d}t = \,\mathrm{d}t\end{aligned}$$

Similarly, one has

$$\begin{aligned}d[Z,B]_t &= (dZ_t)(dB_t) = \rho (dB_t)^2 + \sqrt{1-\rho^2}(\,\mathrm{d}B_t)(dB_t) = \rho\,\mathrm{d}t + 0 = \rho\,\mathrm{d}t \\ d[Z,B]_t &= (dZ_t)(\,\mathrm{d}B_t) = \rho(dB_t)(\,\mathrm{d}B_t) + \sqrt{1-\rho^2}(\,\mathrm{d}B_t)^2 = 0 + \sqrt{1-\rho^2}\,\mathrm{d}t = \sqrt{1-\rho^2}\,\mathrm{d}t\end{aligned}$$

Now let σ_t and g_t be adapted stochastic processes (adapted to $\mathcal{F}_t$) with

$$\int_0^t \sigma_s^2\,\mathrm{d}s < \infty \qquad \text{and} \qquad \int_0^t g_s^2\,\mathrm{d}s < \infty$$

almost surely. Now define

$$\begin{aligned}dM_t &= \sigma_t\,\mathrm{d}B_t & dN_t &= g_t\,\mathrm{d}B_t \\ dU_t &= \sigma_t dB_t & dV_t &= \sigma_t dZ_t\end{aligned}$$

Of course these are just formal expression. For example, $dM_t = \sigma_t\,\mathrm{d}B_t$ means $M_t = M_0 + \int_0^t \sigma_s dB_s$. Using the multiplication table from before we have

$$\begin{aligned}d[M]_t &= (dM_t)^2 = \sigma_t^2(\,\mathrm{d}B_t)^2 = \sigma_t^2\,\mathrm{d}t & d[U]_t &= (dU_t)^2 = \sigma_t^2 (dB_t)^2 = \sigma_t^2\,\mathrm{d}t \\ d[N]_t &= (dN_t)^2 = g_t^2(\,\mathrm{d}B_t)^2 = g_t^2\,\mathrm{d}t & d[V]_t &= (dV_t)^2 = \sigma_t^2(dZ_t)^2 = \sigma_t^2\,\mathrm{d}t\end{aligned}$$

and the cross-quadratic variations

$$\begin{aligned}
d[M,N]_t &= (dM_t)(dN_t) = \sigma_t g_t (\mathrm{d}B_t)^2 = \sigma_t g_t \,\mathrm{d}t \\
d[M,U]_t &= (dM_t)(dU_t) = \sigma_t^2 (\mathrm{d}B_t)(dB_t) = 0 \\
d[M,Z]_t &= (dM_t)(dZ_t) = \rho\sigma_t^2 (\mathrm{d}B_t)(dB_t) + \sqrt{1-\rho^2}\sigma_t^2 (\mathrm{d}B_t)^2 = \sqrt{1-\rho^2}\sigma_t^2 \,\mathrm{d}t
\end{aligned}$$

Next we define

$$dH_t = \mu_t \,\mathrm{d}t \qquad \text{and} \qquad dK_t = f_t \,\mathrm{d}t$$

and observe that since H_t and K_t have finite first variation we have that

$$d[H]_t = (dH_t)^2 = \mu_t^2 (\mathrm{d}t)^2 = 0 \quad \text{and} \quad d[K]_t = (dK_t)^2 = f_t^2 (\mathrm{d}t)^2 = 0$$

Furthermore if $X_t = H_t + M_t$ and $Y_t = K_t + N_t$ then using the previous calculations

$$\begin{aligned}
d[X]_t &= d[X,X]_t = d[H+M,H+M]_t = d[H]_t + d[M]_t + 2d[H,M]_t = \sigma_t^2 \,\mathrm{d}t \\
d[X,Y]_t &= d[H+M,K+N]_t = d[H,K+N]_t + d[M,K+N]_t = d[M,N]_t = \sigma_t g_t \,\mathrm{d}t
\end{aligned}$$

or using the formal algebra

$$\begin{aligned}
d[X]_t &= (dX_t)^2 = \mu_t^2 (\mathrm{d}t)^2 + 2\mu_t\sigma_t (\mathrm{d}t)(\mathrm{d}B_t) + \sigma_t^2 (\mathrm{d}B_t)^2 = 0 + 0 + \sigma_t^2 \,\mathrm{d}t = \sigma_t^2 \,\mathrm{d}t \\
d[X,Y]_t &= (dX_t)(dY_t) = \mu_t f_t (\mathrm{d}t)^2 + \sigma_t f_t (\mathrm{d}t)(\mathrm{d}B_t) + g_t f_t (\mathrm{d}t)(\mathrm{d}B_t) + \sigma_t g_t (\mathrm{d}B_t)^2 \\
&= 0 + 0 + 0 + \sigma_t g_t \,\mathrm{d}t = \sigma_t g_t \,\mathrm{d}t
\end{aligned}$$

CHAPTER 6

Stochastic Differential Equations

1. Definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{\mathcal{T}}$. Let $B_t = (B_1(t), \ldots, B_m(t)) \in \mathbb{R}^m$ be a m-dimensional Brownian motion with $\{B_j(t)\}_{j=1}^m$ a collection of mutually independent Brownian motions such that $B_j(t) \in \mathcal{F}_t$ and for any $0 \leqslant s < t$, $B_j(t) - B_j(s)$ is independent of $\mathcal{F}_s$. Obviously, these conditions are satisfied by the natural filtration $\{\mathcal{F}_t^B\}_{\mathcal{T}}$.

Definition 1.1. *Let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma_j\colon \mathbb{R}^d \to \mathbb{R}^d$ for $i = 1, \ldots, m$ be fixed functions. An equation of the form*

$$dX_t = \mu(X_t)\,\mathrm{d}t + \sum_{i=1}^m \sigma_i(X_t)\,\mathrm{d}B_j(t) \tag{6.1}$$

representing the integral equation

$$X_t = x + \int_0^t \mu(X_s)\,\mathrm{d}s + \sum_{j=1}^m \int_0^t \sigma_j(X_s)\,\mathrm{d}B_j(s)\,. \tag{6.2}$$

*where X_t is an unknown process is a Stochastic Differential Equation (*SDE*) driven by the Brownian motion $\{B_t\}$. The functions $\mu(x)$, $\sigma(x)$ asre called the* drift *and* diffusion *coefficients, respectively.*

It is more compact to introduce the matrix

$$\sigma(x) = \begin{pmatrix} | & & | \\ \sigma_1(x) & \cdots & \sigma_m(x) \\ | & & | \end{pmatrix} \in \mathbb{R}^{m\times d}$$

and write

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t\,. \tag{6.3}$$

There are different concepts of solution for a SDE. The most natural is the one of *strong* solution:

Definition 1.2. *A stochastic process $\{X_t\}$ is a* strong *solution to the* SDE *(6.1) driven by the Brownian motion B_t with (possibly random) initial condition $X_0 \in \mathbb{R}$ if the following holds*

i) *$\{X_t\}$ is adapted to $\{\mathcal{F}_t\}$,*
ii) *$\{X_t\}$ is continuous,*
iii) *$X_t = X_0 + \int_0^t \mu(X_t)\,\mathrm{d}t + \sum_{j=1}^m \int_0^t \sigma_j(X_t)\,\mathrm{d}B_j(t)$ almost surely.*

Remark 1.3. *Often, the choice of Brownian motion in the above definition is implicit. However, it is important that the strong solution of an* SDE *depends on the chosen Brownian motion driving it. A conceptually useful way to restate strong existence, say for all $t \geqslant 0$, is that there exists a measure map $\Phi\colon (t, B) \mapsto X_t(B)$ from $[0,\infty) \times C([0,\infty), \mathbb{R}^d) \to \mathbb{R}^d$ such that $X_t = \Phi(t, B)$ solves (6.2) and X_t is measurable with respect to the filtration generated by B_t.*

Definition 1.4. *We say that a strong solution to (6.1) (driven by a Brownian motion B_t) is* strongly unique *if for any two solutions X_t, Y_t with the same initial condition X_0 of (6.1) we have that*

$$\mathbb{P}\left[X_t = Y_t \text{ for all } t \in [0,T]\right] = 1\,.$$

Remark 1.5. *By definition, the strong solution of a* SDE *is continuous. For this reason, to prove strong uniqueness it is sufficient to prove that two solutions X_t, Y_t satisfy*

$$\mathbb{P}\left[X_t = Y_t\right] = 1 \qquad \text{for all } t \in [0,T]\,.$$

Indeed, assuming that X_t is a version of Y_t, by countable additivity, the set $A = \{\omega\colon X_t(\omega) = Y_t(\omega) \text{ for all } t \in \mathbb{Q}_+\}$ has probability one. By right-continuity (resp. left-continuity) of the sample paths, it follows that X and Y have the same paths for all $\omega \in A$.

2. Examples of SDEs

We now consider a few useful examples of SDEs that have a strong solution.

2.1. Geometric Brownian motion. The geometric Brownian motion, or Black Scholes model in finance, is a stochastic process X_t that solves the SDE

$$\mathrm{d}X_t = \mu X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t,\,, \tag{6.4}$$

where $\mu, \sigma \in \mathbb{R}$ are constants. This model can be used to describe the evolution of the price X_t of a stock, which is assumed to have a mean increase and fluctuations with variance that depend both linearly on the stock price X_t. The coefficients μ, σ are called the percentage drift and percentage volatility, respectively. We see immediately that (6.4) has a solution by Itô's formula: letting $f(x) = \log x$ we have that

$$\mathrm{d}\log X_t = \frac{1}{X_t}(\mu X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t) - \frac{1}{2}\frac{1}{X_t^2}\sigma^2 X_t^2\,\mathrm{d}t = (\mu - \frac{1}{2}\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t\,,$$

Therefore, by integrating and exponentiating, that the solution of the equation reads

$$X_t = X_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right]\,.$$

The uniqueness of this solution will be proven shortly.

2.2. Stochastic Exponential. Let X_t be an Itô process with differential $\mathrm{d}X_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t$. We consider the following SDE:

$$\mathrm{d}U_t = U_t\,\mathrm{d}X_t \tag{6.5}$$

with initial condition $U_0 = u_0 \in \mathbb{R}$. Note that often one chooses $u_0 = 1$. Since the above SDE is analogous to the ODE $df = f\,\mathrm{d}t$ whose solution is given by the exponential function $f(t) = \exp(t)$, the process U_t solving (6.5) is often called the *stochastic exponential* and one writes $X_t = \mathcal{E}(X)_t$. The following result ensures that this process exists and is unique:

Proposition 2.1. *The* SDE *(6.5) has a unique strong solution, given by*

$$U_t = \mathcal{E}(X)_t := U_0 \exp\left[X_t - X_0 - \frac{1}{2}[X]_t\right] = U_0 \exp\left[\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right)\,\mathrm{d}s + \int_0^t \sigma_s\,\mathrm{d}B_s\right]\,. \tag{6.6}$$

Proof. If $u_0 = 0$, then it is immediate by (6.5) that $U_t \equiv 0$ for all $t \geqslant 0$ and it is the only solution. Now suppose $u_0 \neq 0$. We start by proving existence. The proposed solution is clearly adapted, so defining $V_t = X_t - X_0 - \frac{1}{2}[X]_t$ only have to verify that e^{V_t} satisfies (6.5), *i.e.*, $\mathrm{d}(e^{V_t}) = e^{V_t}\,\mathrm{d}X_t$. First, using Itô's formula, we get

$$\mathrm{d}V_t = \mathrm{d}X_t - \frac{1}{2}\,\mathrm{d}[X]_t = \mathrm{d}X_t - \sigma_t^2\,\mathrm{d}t \tag{6.7}$$

$$\mathrm{d}(e^{V_t}) = e^{V_t}\left(\mathrm{d}V_t + \frac{1}{2}\,\mathrm{d}[V]_t\right). \tag{6.8}$$

Note that (6.7) implies $[V]_t = [X]_t$, because the $\int \sigma_t^2 dt$ part is of finite variation. Therefore $\mathrm{d}V_t = \mathrm{d}X_t - \frac{1}{2}\mathrm{d}[V]_t$. Plugging it back to (6.8), we have the desired equality.
We next check that the solution is *unique*. Suppose we have another solution $\tilde{U}$ that satisfies (6.5). Notice that U is nonzero when $u_0 \neq 0$, we can compute $d(\tilde{U}_t/U_t)$

$$\begin{aligned}
\mathrm{d}(\tilde{U}_t/U_t) &= \tilde{U}_t\,\mathrm{d}(1/U_t) + \frac{1}{U_t}\,\mathrm{d}\tilde{U}_t + \mathrm{d}[\tilde{U}, U^{-1}]_t \\
&= \tilde{U}_t\left[-\frac{\mathrm{d}U_t}{U_t^2} + \frac{d[U]_t}{U_t^3}\right] + \frac{\tilde{U}_t\,\mathrm{d}X_t}{U_t} + \mathrm{d}[\tilde{U}, U^{-1}]_t \\
&= \frac{\tilde{U}_t\sigma_t^2\,\mathrm{d}t}{U_t} + \mathrm{d}[\tilde{U}, U^{-1}]_t.
\end{aligned}$$

We need the stochastic differential of U^{-1}.

$$\begin{aligned}
\mathrm{d}(U^{-1}) &= -\frac{\mathrm{d}U_t}{U_t^2} + \frac{\mathrm{d}[U]_t}{U_t^3} = -\frac{\mathrm{d}X_t}{U_t} + \frac{\sigma_t^2\,\mathrm{d}t}{U_t} \\
\therefore\quad \mathrm{d}[\tilde{U}, U^{-1}]_t &= -\sigma_t^2\tilde{U}_t U_t^{-1} dt
\end{aligned}$$

Plugging this back gives $\mathrm{d}(\tilde{U}_t/U_t) = 0$. So the ratio of the two solution stays a constant for all $t \geqslant 0$. Since both solutions start at u_0, we conclude $\mathbb{P}[\tilde{U}_t = U_t, \forall t \geqslant 0] = 1$. ☐

Similarly to the above definition. we introduce the *stochastic logarithm* $X_t = \mathcal{L}(U)_t$ of a process U_t with a stochastic differential $\mathrm{d}U_t = \mu_t'\,\mathrm{d}t + \sigma_t'\,\mathrm{d}B_t'$ and $U_t \neq 0$ as the solution to the following SDE:

$$\mathrm{d}X_t = \frac{\mathrm{d}U_t}{U_t} \text{and} X_t = 0\,. \tag{6.9}$$

Again we have that the solution to the above SDE exists and is unique.

PROPOSITION 2.2. *Under the conditions listed above, the* SDE *(6.9) has a unique solution given by*

$$\mathcal{L}(U)_t = \log\left(\frac{U_t}{U_0}\right) + \int_0^t \frac{\mathrm{d}[U]_t}{U_t^2}\,.$$

Furthermore, as suggested by the framework of stochastic calculus, the above operators are inverse WRT each other:

PROPOSITION 2.3. *If* $u_0 = 1$ *we have* $\mathcal{L}(\mathcal{E}(X))_t = X_t$ *and if* $U_t \neq 0$ *then* $\mathcal{E}(\mathcal{L}(U))_t = U_t$.

PROOF. It is enough to check that $\mathrm{d}X_t = \mathrm{d}(\mathcal{E}(X)_t)/\mathcal{E}(X)_t$ and $\mathrm{d}U_t = \mathcal{L}(U)_t\mathrm{d}\mathcal{L}(U)_t\,.$ ☐

2.3. Linear SDEs. Let $\{\alpha_t\}, \{\beta_t\}, \{\gamma_t\}, \{\delta_t\}$ be given (*i.e.*, independent on X_t) continuous stochastic processes adapted to the filtration $\{\mathcal{F}_t\}$. We consider the family of SDEs given by

$$\mathrm{d}X_t = (\alpha_t + \beta_t X_t)\,\mathrm{d}t + (\gamma_t + \delta_t X_t)\,\mathrm{d}B_t\,. \tag{6.10}$$

We proceed to solve such family of SDEs, which includes as special cases some of the examples treated previously in this course. We do so in two steps:

i) First we consider the case where $\alpha_t = \gamma_t \equiv 0$. In this case we should solve the SDE

$$\mathrm{d}U_t = \beta_t U_t \,\mathrm{d}t + \delta_t U_t \,\mathrm{d}B_t\,,$$

which by Proposition 2.1, choosing $U_0 = 1$ and defining $\mathrm{d}Y = \beta_t\,\mathrm{d}t + \delta_t\,\mathrm{d}B_t$ has the unique solution

$$U_t = \mathcal{E}(Y)_t = \exp\left[\int_0^t \left(\beta_s - \frac{1}{2}\delta_s^2\right)\mathrm{d}s + \int_0^t \delta_s \mathrm{d}B_s\right].$$

ii) We now proceed to consider the full SDE (6.10), and make the ansatz of a separable solution, *i.e.*, assume that $X_t = U_t V_t$ where $\mathrm{d}V_t = a_t\,\mathrm{d}t + b_t\,\mathrm{d}B_t$ for unknown processes $\{a_t\}, \{b_t\}$. Then we compute

$$\begin{aligned}\mathrm{d}X_t &= U_t\,\mathrm{d}V_t + V_t\,\mathrm{d}U_t + \mathrm{d}[U,V](t)\\ &= U_t a_t\,\mathrm{d}t + U_t b_t\,\mathrm{d}B_t + V_t\beta_t U_t\,\mathrm{d}t + V_t\delta_t U_t\,\mathrm{d}B_t + b_t\delta_t U_t\,\mathrm{d}t\\ &= (a_t U_t + b_t\delta_t U_t + \beta_t X_t)\,\mathrm{d}t + (b_t U_t + \delta_t X_t)\,\mathrm{d}B_t\,.\end{aligned}$$

We notice that the above expression coincides with the RHS of (6.10) if

$$a_t = \frac{\alpha_t - \delta_t\gamma_t}{U_t} \qquad \text{and} \qquad b_t = \frac{\gamma_t}{U_t}\,.$$

This uniquely defines the process V_t (whose initial condition is fixed by the fact that $U_0 = 1$ to be $V_0 = X_0$) as

$$V_t = X_0 + \int_0^t \frac{\alpha_s - \delta_s\gamma_s}{U_s}\,\mathrm{d}s + \int_0^t \frac{\gamma_s}{U_s}\,\mathrm{d}B_s\,,$$

which in turn defines the solution to (6.10) as $X_t = U_t \cdot V_t$.

EXAMPLE 2.4. *Letting $a, b \in \mathbb{R}$, consider the following* SDE *on $t \in (0,T)$:*

$$\mathrm{d}X_t = \frac{b - X_t}{T-t}\,\mathrm{d}t + \mathrm{d}B_t \qquad \text{with } X_0 = a\,. \tag{6.11}$$

It is clear that this is a linear SDE *with*

$$\alpha_t = \frac{b}{T-t}\,,\quad \beta_t = \frac{1}{T-t}\,,\quad \gamma_t = 1\,,\quad \delta_t = 0\,.$$

Therefore, the solution to (6.11) is given by

$$X_t = a\left(1 - \frac{t}{T}\right) + b\frac{t}{T} + (T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}s\,. \tag{6.12}$$

Since $\int_0^t (T-s)^{-1}\mathrm{d}s < \infty$ for all $t < T$ the Itô integral in (6.12) is a martingale. Furthermore, as we have proven in Homework 2 it is a gaussian process. Hence, X_t is also a Gaussian process, with $\mathbb{E}[X_t] = a + (a-b)t/T$ and covariance structure

$$\begin{aligned}Cov(X_s, X_t) &= (T-t)(T-s)\,Cov\left(\int_0^{\min(s,t)} \frac{1}{T-q}\,\mathrm{d}B_q, \int_0^{\min(s,t)} \frac{1}{T-q}\,\mathrm{d}B_q + \int_{\min(s,t)}^{\max(s,t)} \frac{1}{T-q}\,\mathrm{d}B_q\right)\\ &= (T - \max(t,s))(T - \min(t,s))\,Var\left(\int_0^{\min(s,t)} \frac{1}{T-q}\,\mathrm{d}B_q\right) = \min(s,t) - \frac{st}{T}\end{aligned}$$

where in the second equality we have used that the covariance of independent random variables is 0. The above expression suggests that the variance of the process X_t is 0 at $t = 0$ and $t = T$, and maximized at $t = T/2$, while the expected value of the process X_t is on the line interpolating between a and b. Hence the name Brownian Bridge: the process above can be seen as a Brownian motion *with initial condition $B_0 = a$ and conditioned on its final value $B_T = b$. Indeed, one can prove (cfr Klebaner, Example 5.11) that*

$$\lim_{t\to T}(T-t)\int_0^t \frac{1}{T-s}\,\mathrm{d}B_s = 0 \quad a.s.\,.$$

3. Existence and Uniqueness for SDEs

In this section we prove a theorem giving sufficient conditions on the coefficients $\mu(\cdot), \sigma(\cdot)$ for the existence and uniqueness of the solution to the associated SDE. We will consider solutions to the equation

$$dX_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t \tag{6.13}$$

Note that we have assumed explicit dependence on time t of the drift and diffusion coefficients as this will allow us to weaken the conditions of the following theorem:

THEOREM 3.1. *Fix a terminal time $T > 0$. Assume that $\sigma(t,x)$ and $\mu(t,x)$ are globally Lipschitz continuous, i.e., that there is a positive constant K so that for any $t \in [0,T]$ and any x and y we have $|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leqslant K|x-y|$. Then the SDE (6.13) with initial condition $X_0 = x$ has a solution*

$$X_t = x + \int_0^t \mu(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s\,,$$

and this solution is strongly unique. Furthermore the solution satisfies is in $L^2(\Omega \times [0,T])$, i.e.,

$$\mathbb{E}\left[\int_0^t X_s^2 \mathrm{d}s\right] < \infty\,.$$

It is clear that in order for the solution X_t to be well defined we need that

$$\int_0^t |\mu(s,X_s)|\,\mathrm{d}s < \infty \qquad \text{and} \qquad \int_0^t \sigma(s,X_s)^2\,\mathrm{d}s < \infty \quad \text{a.s.}$$

However, the assumptions of Theorem 3.1, while being easier to check, are more strict than the ones above. It is useful to recall that these assumptions are needed even for ODEs to have existence and uniqueness: we remind in the following examples how the non-Lipschitz character of the drift can

EXAMPLE 3.2 (Existence). *The* ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 \qquad \textit{with} \qquad x(0) = 1$$

has a drift coefficient $\mu(x) = x^2$ that is not uniformly Lipschitz continuous (although it is locally Lipschitz continuous) because it grows faster than linearly. This ODE *has a solution $x(t) = \frac{1}{1-t}$. However, it is clear that this solution is only well defined for all $t \in (0,1)$ and diverges for $t \to 1$. In other words, the solution to this* ODE *does not exist beyond $t = 1$.*

EXAMPLE 3.3 (Uniqueness). *The* ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\sqrt{|x|}$$

is not locally Lipschitz continuous at $x = 0$. The solution of this ODE *with initial condition $x_0 = 0$ is not unique. Indeed, it is immediate to check that*

$$x_{1,t} = 0 \qquad \text{and} \qquad x_{2,t} = t^2\,,$$

are both solution to this equation with the given initial condition.

3.1. Proof of Theorem 3.1.

Preparatory results. We start by proving two very useful lemmas:

LEMMA 3.4. *(Gronwall's inequality) Let $y(t)$ be a nonnegative function such that*

$$y(t) \leqslant A + D \int_0^t y(s)\,\mathrm{d}s \tag{6.14}$$

for nonnegative $A, D \in \mathbb{R}$. Then $y(t)$ satisfies

$$f(t) \leqslant A \exp\left(Dt\right)$$

PROOF OF LEMMA 3.4. By repeatedly iterating the (6.14) we obtain

$$\begin{aligned}
y(t) &\leqslant A + \int_0^t Dy(s)\,\mathrm{d}s \\
&\leqslant A + \int_0^t D\left(A + \int_0^s Dy(q)\,\mathrm{d}q\right)\,\mathrm{d}s \\
&\leqslant A + A\int_0^t D\,\mathrm{d}s + \int_0^t D \int_0^s Dy(q)\,\mathrm{d}q\,\mathrm{d}s \\
&\leqslant A + ADt + D^2 \int_0^t \int_0^s \left(A + D\int_0^s y(r)\,\mathrm{d}r\right)\,\mathrm{d}q\,\mathrm{d}s \\
&\leqslant A + ADt + AD^2 \int_0^t \int_0^s \mathrm{d}q\,\mathrm{d}s + D^3 \int_0^t \int_0^s \int_0^q y(r)\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}s \leqslant \dots \\
&\leqslant A + ADt + A\frac{D^2t^2}{2} + A\frac{D^3t^3}{3!} + D^4 \int_0^t \int_0^s \int_0^q \int_0^r y(\tau)\,\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}s\,.
\end{aligned}$$

We notice that repeating the above procedure k times we will obtain the first k terms of the Taylor expansion of $A \exp Dt$ plus a remainder term resulting from an integral iterated $k+1$ times. For finite T we can bound such integral by defining the constants

$$C := \int_0^T y(s)\,\mathrm{d}s < \infty \qquad \text{and} \qquad G := A + DC\,,$$

so that $y(t) \leqslant G$. Consequently, we can bound the remainder term by $Gt^{k+1}D^{k+1}/k!$ which vanishes exponentially fast in the limit $k \to \infty$, uniformly in $t \in [0, T]$.

Alternative proofs assuming the existence and uniqueness of solutions to ODEs can be found in any good ODE or dynamics book. For instance [**6**] or [**5**]. ☐

LEMMA 3.5. *Let $\{y_n(t)\}$ be a sequence of functions satisfying*

- $y_0(t) \leqslant A\,,$
- $y_{n+1}(t) \leqslant D \int_0^t y_n(s)\,\mathrm{d}s < \infty \quad \forall\, n > 0, t \in [0, T]\,,$

for positive constants $A, D \in \mathbb{R}$, then $y_n(t) \leqslant CD^n t^n/n!$.

PROOF. the proof of this result goes by induction: the first step is trivial, while for the induction step we have

$$y_{n+1}(t) \leqslant D \int_0^t y_n(s)\,\mathrm{d}s \leqslant D \int_0^t C\frac{D^n t^n}{n}!\,\mathrm{d}s = C\frac{D^{n+1}t^{n+1}}{(n+1)!}\,.$$

☐

Uniqueness for SDE. If $X_1(t)$ and $X_2(t)$ are two solutions then taking there difference produces

$$X_1(t) - X_2(t) = \int_0^t [\mu(s, X_1(s)) - \mu(s, X_2(s))] \,\mathrm{d}s + \int_0^t [\sigma(s, X_1(s)) - \sigma(s, X_2(s))] \,dB_s$$

We now use the fact that

$$\max\{(a-b)^2, (a+b)^2\} \leqslant (a-b)^2 + (a+b)^2 = 2a^2 + 2b^2$$

gives

$$\begin{aligned} |X_1(t) - X_2(t)|^2 \leqslant & 2\left|\int_0^t [\mu(s, X_1(s)) - \mu(s, X_2(s))] \,\mathrm{d}s\right|^2 + 2\left|\int_0^t [\sigma(s, X_1(s)) - \sigma(s, X_2(s))] \,dB_s\right|^2 \\ \leqslant & (I) + (II) \end{aligned}$$

Next recall that Holders (or Cauchy-Schwartz) inequality implies that $(\int_0^t f \,\mathrm{d}s)^2 \leqslant t \int_0^t f^2 \,\mathrm{d}s$ (apply Holder to the product $1 \cdot f$ with $p = q = 2$.) Hence

$$\mathbb{E}(I) \leqslant 2t\mathbb{E} \int_0^t [\mu(s, X_1(s)) - \mu(s, X_2(s))]^2 \,\mathrm{d}s \leqslant 2tK^2 \int_0^t \mathbb{E}\,|X_1(s) - X_2(s)|^2 \,\mathrm{d}s$$

Applying Itô's isometry to the second term gives

$$\mathbb{E}(II) = 2\int_0^t \mathbb{E}\,|\sigma(s, X_1(s)) - \sigma(s, X_2(s))|^2 \,\mathrm{d}s \leqslant 2K^2 \int_0^t \mathbb{E}\,|X_1(s) - X_2(s)|^2 \,\mathrm{d}s$$

Putting this all together and recalling that $t \in [0, T]$ gives

$$\mathbb{E}\,|X_1(t) - X_2(t)|^2 \leqslant 2K^2(T+1) \int_0^t \mathbb{E}\,|X_1(s) - X_2(s)|^2 \,\mathrm{d}s$$

Hence by Gronwall's inequality Lemma 3.4 we conclude that $\mathbb{E}\,|X_1(t) - X_2(t)|^2 = 0$ for all $t \in [0, T]$. Hence $X_1(t)$ and $X_2(t)$ are identical almost surely.

Existence for SDE. The existence of solutions is proved by a variant of Picard's iterations. Fix an initial value x, we define a sequence processes $X_n(t)$ follows. By induction, the processes have continuous paths and are adapted.

$$\begin{aligned} X_0(t) &= x \\ X_1(t) &= x + \int_0^t \mu(s, x) ds + \int_0^t \sigma(s, x) dB_s \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ X_{n+1}(t) &= x + \int_0^t \mu(s, X_n(s)) ds + \int_0^t \sigma(s, X_n(s)) dB_s \end{aligned}$$

Fix $t \geqslant 0$, we will show that $X_n(t)$ converges in L^2. Hence there is a random variable $X(t) \in L^2(\Omega, \mathcal{F}, P)$ and $X_n \xrightarrow{L^2} X(t)$. Let $y_n(t) = \mathbb{E}\left[(X_{n+1}(t) - X_n(t))^2\right]$, we will verify the two conditions in Lemma 3.5. First, for $n = 0$ and any $t \in [0, T]$,

$$\begin{aligned} y_0(t) = \mathbb{E}\left[(X_1(t) - X_0(t))^2\right] &\leqslant 2\mathbb{E}\left[\left(\int_0^t \mu(s, x) ds\right)^2\right] + 2\mathbb{E}\left[\left(\int_0^t \sigma(s, x) dB_s\right)^2\right] \\ &\leqslant 2\mathbb{E}\left[\left(\int_0^t K|1 + x| ds\right)^2\right] + 2\mathbb{E}\left[\left(\int_0^t K|1 + x| dB_s\right)^2\right] \leqslant C < \infty, \end{aligned}$$

where the second inequality uses the fact that the coefficients are growing no faster than linearly. Second, similar computation as for the uniqueness yields

$$y_{n+1}(t) \leqslant 2K^2(1+T)\int_0^t y_n(s)ds \quad \forall t \in [0,T],\ n = 0,1,2\dots$$

which is finite by induction. Lemma 3.5 implies

$$y_n(t) = \mathbb{E}\left[(X_{n+1}(t) - X_n(t))^2\right] \leqslant C\frac{(4K^2+4K^2T)^n}{n!},$$

which goes to zero uniformly for all $t \in [0,T]$. We thus conclude $X_n(t)$ converges in L^2 uniformly and their L^2-limit, $X(t) \in L^2(\Omega, \mathcal{F}, P)$.

It remains to show that the limit process $X(t)$ solves (6.13). Since $X_n \xrightarrow{L^2} X$, we have

$$\begin{aligned}&\mathbb{E}[\mu(t,X_n(t)) - \mu(t,X(t))]^2 + \mathbb{E}[\sigma(t,X_n(t)) - \sigma(t,X(t))]^2\\ &\leqslant K^2\mathbb{E}[(X_n(t)-X(t))^2] + K^2\mathbb{E}[(X_n(t)-X(t))^2]\\ &\to 0, \quad \text{uniformly in } t\end{aligned}$$

By Itô's isometry and Fubini:

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t \sigma(s,X_n(s))dB_s - \int_0^t \sigma(s,X(s))dB_s\right)^2\right] &= \mathbb{E}\left[\left(\int_0^t \sigma(s,X_n(s)) - \sigma(s,X(s))dB_s\right)^2\right]\\ &= \int_0^t \mathbb{E}\left[(\sigma(s,X_n(s)) - \sigma(s,X(s)))^2\right]\mathrm{d}s \overset{n\to\infty}{\longrightarrow} 0\,.\end{aligned}$$

Similarly, by Cauchy-Schwarz inequality we have that :

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t \mu(s,X_n(s))dB_s - \int_0^t \mu(s,X(s))\,\mathrm{d}s\right)^2\right] &= \mathbb{E}\left[\left(\int_0^t \mu(s,X_n(s)) - \mu(s,X(s))\,\mathrm{d}s\right)^2\right]\\ &= t\int_0^t \mathbb{E}\left[(\mu(s,X_n(s)) - \mu(s,X(s)))^2\right]\mathrm{d}s \overset{n\to\infty}{\longrightarrow} 0\,.\end{aligned}$$

We thus have

$$X(t) = x + \int_0^t \mu(s,X(s))ds + \int_0^t \sigma(s,X(s))dB_s,$$

i.e., $X(t)$ solves (6.13).

Remark 3.6. *Looking through the proof of Theorem 3.1 we see that the assumption of global Lipschitz continuity can be weakened to the following assumption*

i) $|\mu(t,x)| + |\sigma(t,x)| < C(1+|x|)$ *(necessary for existence)*,
ii) $|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leqslant C|x-y|$ *(necessary for uniqueness)*.

4. Weak solutions to SDEs

Until now we have studied strong solutions to SDEs, *i.e.*, solutions for which a Brownian motion (and a probability space) is given *in advance*, and that are constructed based on such Brownian motion. If we are only given some functions $\mu(x)$ and σ *without* fixing a Brownian motion, we may be able to construct a *weak solution* to an SDE of the form (6.1). Such solutions allow to choose a convenient Brownian motion (and consequently a probability space!) for the solution X_t to satisfy the desired SDE.

Definition 4.1. *A weak solution of the stochastic differential equation (6.1) with initial condition* $X_0 \sim \rho_0$ *for a given probability distribution* ρ_0 *is a continuous stochastic process* X_t *defined on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that for* some *Brownian motion* B_t *and* some *filtration* $\{\mathcal{F}_t\}$ *such that* $B_j(t) \in \mathcal{F}_t$ *and for any* $0 \leqslant s < t$, $B_j(t) - B_j(s)$ *is independent of* $\mathcal{F}_s$, *the process* X_t *is adapted and satisfies the stochastic integral equation (6.2).*

In other words, in the case of a weak solution we are free to chose *some* convenient Brownian motion that allows X_t to be a solution. In this sense, these solutions are also distributional solutions, *i.e.*, solutions that have the "right" marginals.

Because two solutions X_t, Y_t may live on different probability spaces, we cannot compare their paths as in the case of strong solutions. Instead, we weaken the concept of strong uniqueness to the one of *weak uniqueness*, *i.e.*, uniqueness in law of the solution process:

Definition 4.2. *The weak solution of a* SDE *is said to be* weakly unique *if any two solutions* X_t, Y_t *have the same law, i.e., for all* $\{t_i \in [0, T]\}$, $\{A_i \in \mathcal{B}\}$ *we have*

$$\mathbb{P}\left[\bigcap_i \{X_{t_i} \in A_i\}\right] = \mathbb{P}\left[\bigcap_i \{Y_{t_i} \in A_i\}\right].$$

Example 4.3. *Consider the* SDE $\mathrm{d}Y_t = \mathrm{d}B_t$ *with initial condition* $Y_0 = 0$. *This* SDE *has clearly a strong solution, which is* $Y_t = B_t$. *If we let* W_t *be* another *Brownian motion (possibly defined on another probability space) then* W_t *will* not *be, in general, a strong solution to the above* SDE *(in the case that the two probability spaces are different, the two solutions cannot even be compared). It will, however, be a* weak *solution to the* SDE, *as being a Brownian motion completely determines the marginals of the process.*

We will now consider an example for which there exists a weak solution, but not a strong solution:

Example 4.4 (Tanaka's SDE). *For certain* μ *and* σ, *solutions to* (6.1) *may exist for some Brownian motion and some admissible filtrations but not for others. Consider the* SDE

$$dX_t = sign(X_t) dB_t, \quad X_0 = 0; \tag{6.15}$$

where $\sigma(t, x) = sign(x)$ *is the sign function*

$$sign(x) = \begin{cases} +1, & \text{if } x \geqslant 0 \\ -1, & \text{if } x < 0. \end{cases}$$

The function $\sigma(x)$ *is not continuous and thus not Lipschitz. A strong solution does not exist for this* SDE, *with the filtration* $\mathbb{F} = (\mathcal{F}_t)$ *chosen to be* $\mathcal{F}_t := \sigma(B_s, 0 \leqslant s \leqslant t)$. *Suppose* X_t *is a strong solution to Tanaka's* SDE, *then we must have*

$$\tilde{\mathcal{F}}_t := \sigma(X_s, 0 \leqslant s \leqslant t) \subseteq \mathcal{F}_t. \tag{6.16}$$

Notice that for any $T \geqslant 0$, $\int_0^T \mathbb{E}\left[sign(X_t)^2\right] ds < \infty$, *the Itô integral* $\int_0^t sign(X_t) dB_s$ *is well defined and* X_t *is a martingale. Moreover, the quadratic variation of* X_t *is*

$$[X]_t = \int_0^t [sign(X_t)]^2 ds = \int_0^t 1 \cdot ds = t,$$

thus X_t *must be a Brownian motion (by Lévy's characterization, to be proved later). We may denote* $X_t = \tilde{B}_t$ *to emphasize that it is a Brownian motion. Now multiplying both sides of* (6.15) *by* $sign(X_t)$, *we obtain*

$$dB_t = \mathrm{sign}(\tilde{B}_t) d\tilde{B}_t. \tag{6.17}$$

and thus $B_t = \int_0^t \operatorname{sign}(\tilde{B}_s) d\tilde{B}_s$. *By Tanaka's formula (to be shown later), we then have*

$$B_t = |\tilde{B}_t| - \tilde{L}_t$$

where $\tilde{L}_t$ *is the local time of* $\tilde{B}_t$ *at zero. It follows that* B_t *is* $\sigma(|\tilde{B}_s|, 0 \leqslant s \leqslant t)$*-measurable. This leads to a contradiction to* (6.16)*, because it would imply that*

$$\mathcal{F}_t \subseteq \sigma(|\tilde{B}_s|, 0 \leqslant s \leqslant t) \subsetneq \sigma(\tilde{B}_s, 0 \leqslant s \leqslant t) = \tilde{\mathcal{F}}_t.$$

Still, as we have seen above, choosing $X_t = \tilde{B}_t$ *there exists a Brownian motion* B_s *such that Tanaka's* SDE *holds. Such pair of Brownian motions forms a weak solution to Tanaka's equation.*

5. Markov property of Itô diffusions

The solutions (weak or strong) to stochastic differential equations are referred to as *diffusion processes* (or Itô diffusions).

DEFINITION 5.1. *An Itô process* $\mathrm{d}X_t = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}B_t$ *is an Itô diffusion if* μ_t, σ_t *are measurable* WRT *the filtration* $\{\mathcal{F}_t^X\}$ *generated by* X_t *for all* $t \in [0, T]$*, i.e.,*

$$\mu_t, \sigma_t \in \mathcal{F}_t^X \,.$$

REMARK 5.2. *It is clear that solution* $\{X_t\}$ *to the* SDE $\mathrm{d}X_t = \mu(t, X_t) \,\mathrm{d}t + \sigma(t, X_t) \,\mathrm{d}B_t$ *for continuous functions* μ, σ *is an Itô diffusion. For this reason, such* SDE*s are called of* diffusion-type*.*

Recall Def. 7.12 that a Markov process is a process whose future depends on its past only through its present value, while if this property holds also for stopping times, the process is said to have the strong Markov property (CFR Def. 7.16).

THEOREM 5.3. *The solution* $\{X_t\}$ *to the* SDE

$$\mathrm{d}X_t = \mu(t, X_t) \,\mathrm{d}t + \sigma(t, X_t) \,\mathrm{d}B_t \,, \tag{6.18}$$

has the strong Markov property.

While we do not present the proof of this result, which can be found in [**14**], it should be intuitively clear why solutions to (6.18) have the Markov property. Indeed, we see that the drift and diffusion coefficients of the above SDE only depend on the time and on the value of X_t at that time (and not on its past value). This fact, combined with the independence of the increments of Brownian motion results in the Markov (and the strong Markov) property of such solutions.

CHAPTER 7

PDEs and SDEs: The connection

Throughout this chapter, except when specified otherwise, we let $\{X_t\}$ be a solution to the SDE

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t\,. \tag{7.1}$$

As we have seen in the last chapter, solutions to the above equation are referred to as diffusion processes. This name comes from the fact that Brownian motion, the archetypal diffusion process, was invented to model the diffusion of a dust particle in water. Similarly, in the world of partial differential equations, diffusion equations model precisely the same type of phenomenon. In this chapter we will see that the correspondence between these two domains goes well beyond this point.

1. Infinitesimal generators

Having seen in the previous chapter that solutions to SDEs possess the strong markov property, we introduce the following operator to study the evolution of their finite-dimensional distributions:

DEFINITION 1.1. *The* infinitesimal generator *for a continuous time Markov process X_t is an operator A such that for any function f,*

$$A_t f(x) := \lim_{dt\downarrow 0} \frac{\mathbb{E}[f(X_{t+dt})|X_t = x] - f(x)}{dt}, \tag{7.2}$$

provided the limit exists. The set of function for which the above limit exists is called the domain *$\mathcal{D}(A_t)$ of the generator.*

This operator encodes the infinitesimal change in the probability distribution of the process X_t. One way of seeing this is by choosing $f(x) = \mathbb{1}_A(x)$ for a set $A \in \mathbb{R}^d$.

We now look at some examples where we find the explicit form of the generator for Itô diffusions:

EXAMPLE 1.2. *The infinitesimal generator for a standard one-dimensional Brownian motion B_t is*

$$A = \frac{1}{2}\frac{d^2}{dx^2}$$

for all f that are C^2 with compact support. To derive this, we first apply Itô's formula to any $f \in C^2$ and write

$$\begin{aligned} f(B_t) &= f(B_0) + \int_0^t \frac{d^2}{dx} f(B_s) dB_s + \int_0^t \frac{1}{2}\frac{d^2}{dx^2} f(B_s) ds \\ &= f(B_0) + \int_0^t f'(B_s) dB_s + \int_0^t \frac{1}{2}\frac{d^2}{dx^2} f(B_s) ds \end{aligned}$$

Apply this formula to two time points t and $t + r$, we have

$$f(B_{t+r}) = f(B_t) + \int_t^{t+r} f'(B_s) dB_s + \int_t^{t+r} \frac{1}{2}\frac{d^2}{dx^2} f(B_s) ds$$

When f has compact support, $f'(x)$ is bounded and suppose $|f'(x)| \leqslant K$ and thus for each t,

$$\int_0^t \mathbb{E}\left(f'^2(B_s)\right) ds \leqslant \int_0^t K^2 ds = K^2 t < \infty.$$

Hence the first integral has expectation zero, due to Itô's isometry. It follows that

$$\begin{aligned}
\mathbb{E}[f(B_{t+r})|B_t = x] &= f(x) + \mathbb{E}\left\{ \int_t^{t+r} f'(B_s)dB_s + \int_t^{t+r} \frac{1}{2}\frac{d^2}{dx^2}f(B_s)ds \middle| B_t = x \right\} \\
&= f(x) + \mathbb{E}\left\{ \int_t^{t+r} f'(B_s)dB_s + \int_t^{t+r} \frac{1}{2}\frac{d^2}{dx^2}f(B_s)ds \right\} \\
&= f(x) + \mathbb{E}\int_t^{t+r} \frac{1}{2}\frac{d^2}{dx^2}f(B_s)ds
\end{aligned}$$

where the second equality is due to the independence of the post-t process $B_{t+s} - B_t$ and B_t. Subtracting $f(x)$, dividing by r and letting $r \to 0$ on both sides, we obtain

$$\begin{aligned}
Af(x) &= \lim_{r\downarrow 0} \frac{\mathbb{E}[f(B_{t+r})|B_t = x] - f(x)}{r} = \lim_{r\downarrow 0} \frac{\mathbb{E}\left[\int_t^{t+r} \frac{1}{2}\frac{d^2}{dx^2}f(B_s)ds\right]}{r} \\
&= \frac{d\left(\mathbb{E}\left[\int_t^{t+r} \frac{1}{2}\frac{d^2}{dx^2}f(B_s)ds\right)\right]}{dr} \overset{(*)}{=} \frac{d\int_t^{t+r} \frac{1}{2}\mathbb{E}\left[\frac{d^2}{dx^2}f(B_s)\right]ds}{dr} \\
&= \frac{1}{2}\frac{d^2}{dx^2}f(B_s)\bigg|_{s=t} = \frac{1}{2}\frac{d^2}{dx^2}f(B_t) = \frac{1}{2}\frac{d^2}{dx^2}f(x)
\end{aligned}$$

In the above calculation, we inverted the order of the integrals using Fubini-Tonelli's theorem.

REMARK 1.3. *Here we omit the subscript t in the generator A because Brownian motion is time-homogeneous, i.e.,*

$$\frac{\mathbb{E}[f(B_{t+dt})|B_t = x] - f(x)}{dt} = \frac{\mathbb{E}[f(B_{s+dt})|B_s = x] - f(x)}{dt}$$

and thus $A_t f(x) = A_s f(x)$. The generator $A = \frac{1}{2}\frac{d^2}{dx^2}$ does not change with time.

The procedure to obtain the infinitesimal generator of Brownian motion can be straightforwardly generalized to the case of Itô diffusions:

EXAMPLE 1.4. *Assume that X_t satisfies the SDE (7.1), then its generator A_t is*

$$A_t f(x) = \mu(t,x)\frac{d}{dx}f(x) + \frac{\sigma^2(t,x)}{2}\frac{d^2}{dx^2}f(x) \tag{7.3}$$

for all $f \in C^2$ with compact support.The computation is similar to the Brownian motion case. First apply Itô's formula to $f(X_t)$ and get

$$f(X_{t+r}) = f(X_t) + \int_t^{t+r} \left\{\mu(s,X_s)\frac{d}{dx}f(X_s) + \frac{\sigma^2(s,X_s)}{2}\frac{d^2}{dx^2}f(X_s)\right\} ds + \int_t^{t+r} \sigma(s,X_s)\frac{d}{dx}f(X_s)dB_s$$

Then using the fact that $f \in C^2$ with compact support, the last integral has expectation zero. Conditioning on $X_t = x$, computing $\frac{E[f(X_{t+r})|X_t=x]-f(x)}{r}$, exchanging integrals by Fubini-Tonelli and taking $r \to 0$, we conclude that the generator has the form (7.3).

The above example can be further generalized to the case when the function f also depends on time:

Example 1.5. *Consider the two dimensional process (t, X_t), where the first coordinate is deterministic and the second coordinate X_t satisfies (7.1). We treat it as a process $Y_t = (t, X_t) \in \mathbb{R}^2$. In this case, the generator of Y_t, according to the definition in (7.2) is given by*

$$\begin{aligned} A_t f(t,x) &:= \lim_{dt \downarrow 0} \frac{\mathbb{E}[f(Y_{t+dt})|Y_t = (t,x)] - f(t,x)}{dt} \\ &= \mu(t,x)\frac{\partial}{\partial x} f(t,x) + \frac{\sigma^2(t,x)}{2}\frac{\partial^2}{\partial x^2} f(t,x) + \frac{\partial}{\partial t} f(t,x) \end{aligned} \tag{7.4}$$

for any $f \in C^{1,2}$ that has compact support.

Formally speaking, if A_t is the generator of X_t, what A_t does to f is to map it to the "*drift coefficient*" in the stochastic differential of $f(X_t)$, i.e.,

$$df(X_t) = A_t f(X_t) \cdot dt + (\text{something}) \cdot dB_t$$

Remark 1.6. *The notation here is slightly different from [Klebaner], where Klebaner always uses L_t to denote the operator on functions $f \in C^{1,2}$ so that*

$$L_t f(t,x) = \mu(t,x)\frac{\partial}{\partial x} f(t,x) + \frac{\sigma^2(t,x)}{2}\frac{\partial^2}{\partial x^2} f(t,x) \tag{7.5}$$

and call such L_t the "generator of X_t". Comparing this form with (7.3) and (7.4), and since L_t acts on $C^{1,2}$ functions, we can relate L_t to the generator A_t of (t, X_t), i.e.,

$$L_t f(t,x) + \frac{\partial}{\partial t} f(t,x) = A_t f(t,x)$$

When we look at martingales constructed from the generators, A_t will give a more compact (and maybe more intuitive) notation.

Exercise. Find the generator A_t of (X_t, Y_t), where X_t and Y_t satisfies

$$\begin{aligned} dX_t &= \mu(t, X_t)dt + \sigma(t, X_t)dB_t \\ dY_t &= \alpha(t, Y_t)dt + \beta(t, Y_t)dB_t \end{aligned}$$

What if X_t and Y_t are driven by two independent Brownian motions?

2. Martingales associated with diffusion processes

Suppose X_t solves (7.1) and A_t is its generator (see (7.3)). For $f \in C^2$, we know from Itô's formula that

$$f(X_t) = f(X_0) + \int_0^t A_s f(X_s)dt + \int_0^t \sigma(s, X_s)f'(X_s)dB_s$$

Under proper conditions, the third term on the right is a well-defined Itô integral and also a martingale. We can construct martingales by isolating this integral, i.e., let

$$M_t := f(X_t) - f(X_0) - \int_0^t A_s f(X_s)ds \quad \left(= \int_0^t \sigma(s, X_s)f'(X_s)dB_s\right), \tag{7.6}$$

and we will see that for certain μ, σ and f functions, M_t will be a martingale. First of all, if X_t is a solution to (7.1) (either weak or strong), then, by definition, M_t is always $\mathcal{G}_t := \sigma(X_s, 0 \leqslant s \leqslant t)$ measurable. So from now on, for the purpose of constructing martingales, we will only say "X_t solves the SDE (7.1)" without specifying whether X_t is a strong or a weak solution. Recall that if

$$\int_0^t \mathbb{E}[\sigma^2(s, X_s)f'^2(X_s)]\, ds < \infty, \tag{7.7}$$

then the Itô integral $\int_0^t \sigma(s, X_s) f'(X_s) dB_s$ is a martingale. Therefore, the usual technical step is to prove (7.7) in order to conclude that M_t is a martingale.
Theorems 6.2 in [Klebaner] gives a set of conditions for M_t to be a martingale:

CONDITION 1. *Let the following assumptions hold:*

i) *$\mu(t, x)$ and $\sigma(t, x)$ are locally Lipschitz in x with a Lipschitz constant independent of t and are growing at most linearly in x; and*

ii) *$f \in C^2$ and f' is bounded,*

Condition (i) implies that (7.1) has a strong solution, but more importantly, it controls the speed of growth of the solution X_t (see Theorem 5.4 and also the proof in Theorem 6.2 of [Klebaner] for more details); (ii) controls the magnitude of f, which together with (i) ensure the finiteness of the integral in (7.7). [**9**, Theorem 6.3] gives a n alternative set of conditions to Condition 1, however, the proof follows the same idea. The above result can be summarized in the following theorem

THEOREM 2.1. *Let $\{X_t\}$ be a solution to (7.1), f a function such that Condition 1 holds, then M_t defined in (7.6) is a martingale.*

We now generalize the above result to the case when f is time-dependent. Let X_t solve (7.1). If A_t is the generator of the two dimensional process (t, X_t) (see the expression in (7.4)), then for any function $f(t, x) \in C^{1,2}$

$$M_t := f(t, X_t) - f(0, X_0) - \int_0^t A_s f(s, X_s) ds \tag{7.8}$$

can be a martingale if μ, σ and f satisfy certain conditions. Again, using Itô's formula, we see that

$$M_t = \int_0^t \sigma(s, X_s) \frac{\partial}{\partial x} f(s, X_s) dB_s$$

The approach to show that M_t is a martingale is the same as above. For example, if Condition 1 (ii) above is modified to

CONDITION 2. *Let the following assumptions hold:*

i) *$\mu(t, x)$ and $\sigma(t, x)$ are locally Lipschitz in x with a Lipschitz constant independent of t and are growing at most linearly in x; and*

ii)' *$f \in C^{1,2}$ and $\frac{\partial}{\partial x} f(t, x)$ is bounded for all t and x.*

then, we can conclude that M_t defined in (7.8) is a martingale:

THEOREM 2.2. *Let $\{X_t\}$ be a solution to (7.1), f a function such that Condition 2 holds, then M_t defined in (7.8) is a martingale.*

One advantage of using A_t instead of L_t is that we can express M_t in the same form, that is, $M_t := f(X_t) - f(X_0) - \int_0^t A_s f(X_s) ds$, provided A_t is chosen to be the generator of X_t (which might be high-dimensional). The following are a few immediate consequences, stated under Condition 2. However, one should keep in mind that there are other conditions, under which these claims are also true.

COROLLARY 2.3 (Dynkin's formula). *Suppose that X_t solves* (7.1) *and that Condition 2 holds. Let A_t be the generator of* (t, X_t) *(see* (7.4)*). Then for any $t \in [0, T]$,*

$$\mathbb{E}[f(t, X_t)] = f(0, X_0) + \mathbb{E}\left[\int_0^t A_s f(s, X_s) ds\right],$$

The result is also true if t is replaced by a bounded stopping time $\tau \in [0, T]$.

Corollary 2.4. *Assume that X_t solves* (7.1) *and that Condition 2 holds. If f solves the following* PDE

$$(A_t f(t,x) =) \quad \mu(t,x)\frac{\partial}{\partial x}f(t,x) + \frac{\sigma^2(t,x)}{2}\frac{\partial^2}{\partial x^2}f(t,x) + \frac{\partial}{\partial t}f(t,x) \equiv 0,$$

then $f(t,X_t)$ is a martingale.

Example 2.5. *Consider $X_t = B_t$, then $\sigma \equiv 1$ and $\mu \equiv 0$, which satisfies Condition 2 (i). Then, for any $f(t,x)$ that satisfies Condition 2 (ii') (or conditions in Theorem 6.3 of Klebaner) and solves*

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}f(t,x) + \frac{\partial}{\partial t}f(t,x) \equiv 0,$$

$f(t,B_t)$ is a martingale. For example, $f(t,x) = x, x^2 - t, x^3 - 3tx$, $e^{t/2}\sin(x)$, or $e^{x-t/2}$.

3. Connection with PDEs

In the previous section we have seen that under proper conditions the solution f of some PDE can be used to construct a martingale. In this section, we will see that the solutions of certain PDEs may be represented by the expectation of the solution of the SDE.
Throughout this section we will assume that X_t solves the SDE (7.1) whose coefficients satisfy Condition 2 (i), and A_t is the generator of (t, X_t) as given in (7.4). Furthermore, we assume that f satisfies Condition 2 (ii'). Note that other conditions, under which M_t defined in (7.8) is a martingale, would also work.

3.1. Kolmogorov Backwards Equation.

Theorem 3.1. *Under the standing assumptions, if $f(t,x)$ solves the* PDE

$$\begin{cases} A_t f = 0 & \text{for all } t \in (0,T) \\ f(T,x) = g(x) & \end{cases}, \tag{7.9}$$

for some function g such that $\mathbb{E}[|g(X_T)|] < \infty$. Then,

$$f(t,x) = \mathbb{E}(g(X_T)|X_t = x), \quad \text{for all } t \in [0,T].$$

Proof. Under the standing assumptions and the fact that $A_t f = 0$, we know that $f(t,X_t)$ is a martingale, due to Corollary 2.4. Then for any $t \in [0,T]$,

$$\mathbb{E}[f(T,X_T)|\mathcal{F}_t] = f(t,X_t)$$

Using the boundary condition, we have $f(T,X_T) = g(X_T)$. The result then follows from the Markov property of the solution to the diffusion-type SDE, i.e.,

$$f(t,X_t) = \mathbb{E}[g(X_T)|\mathcal{F}_t] = \mathbb{E}[g(X_T)|X_t].$$

□

Remark 3.2. *Note that the above theorem, assuming that $f(t,x)$ solves the given* PDE*, represents expectation values of the process X_t in terms of such solutions. Under suitable regularity conditions on the coefficients of the* SDE *(7.1) and on the boundary condition $g(x)$ one can show that such expected value is the unique solution to the* PDE *(7.9). These results, however, go beyond the scope of this course and will not be presented here. We refer the interested reader to,* e.g., [**14**].

Definition 3.3. *For a Itô diffusion $\{X_t\}$ solving (7.1) the* PDE *(7.9) is called the* Kolmogorov Backwards equation.

The name of the above PDE is due to the fact that it has to be solved *backwards* in time, *i.e.*, the boundary condition in (7.9) is fixed at the *end* of the time interval of interest. This may seem at first counterintuitive. One possible way to interpret this fact is that bringing the time derivative on the other side of the equality we obtain $-\partial_t f = L_t f$, where L_t is the generator defined in (7.5). In this form, the "arrow of time" is given by the fact that σ^2 has nonnegative values and corresponds to the second derivative "widening the support" of f, and the negative sign in front of the time derivative corresponds to an evolution in the "reverse" direction. Another way of understanding the fact that the direction of time is reversed in (7.9) is the following: in order to establish an expectation WRT a certain function *in the future* (*e.g.*, the value of an option at expiration), an operator that evolves such function should project it *backwards* to the information we have at the moment, *i.e.*, the value of X_t

EXAMPLE 3.4. *Letting $g(x) = \mathbb{1}_A(x)$, we have that being able to solve (7.9) is equivalent to knowing*

$$\mathbb{E}\left[\mathbb{1}_A(X_T)|X_t = x\right] = \mathbb{P}\left[X_T \in A|X_t = x\right] .$$

EXAMPLE 3.5. *Letting X_t be the solution to the Black Scholes model $\mathrm{d}X_t = \mu X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t$ and $g(x) = V(x)$ some value function of an option at time T, then being able to solve (7.9) is equivalent to knowing the expected value of that option at expiration: $\mathbb{E}\left[V(X_T)|X_t = x\right]$.*

We now state an extension of Theorem 3.1 which deals with the case where the right side of the PDE is nonzero.

THEOREM 3.6. *Under the standing assumptions, if $f(t,x)$ solves the PDE*

$$\begin{cases} A_t f(t,x) = -\phi(x) & \text{for all } t \in (0,T) \\ f(T,x) = g(x) \end{cases}$$

for some bounded function $\phi : \mathbb{R} \to \mathbb{R}$ and g such that $\mathbb{E}[|g(X_T)|] < \infty$. Then,

$$f(t,x) = \mathbb{E}\left(g(X_T) + \int_t^T \phi(X_s)ds \,\middle|\, X_t = x\right), \quad \text{for all } t \in [0,T].$$

PROOF. By Theorem 2.2 we have that

$$M_t := f(t,X_t) - f(0,X_0) - \int_0^t A_s f(s,X_s)ds$$

is a martingale. Plugging in $A_t f(t,x) = -\phi(x)$ and taking conditional expectation of $M_T|\mathcal{F}_t$, since $M_t = \mathbb{E}(M_T|\mathcal{F}_t)$ we get

$$f(t,X_t) - f(0,X_0) - \int_0^t A_s f(s,X_s)ds = \mathbb{E}(f(T,X_T)|\mathcal{F}_t) - f(0,X_0) - \mathbb{E}\left(\int_0^T A_s f(s,X_s)ds \,\middle|\, \mathcal{F}_t\right)$$

which can be rewritten as

$$f(t,X_t) = \mathbb{E}\left[g(X_T) + \int_t^T \phi(X_s)ds \,\middle|\, \mathcal{F}_t\right]$$

Finally, by the Markov property of Itô diffusions, we obtain

$$f(t,x) = \mathbb{E}\left[g(X_T) + \int_t^T \phi(X_s)ds \,\middle|\, X_t = x\right]$$

□

3.2. Feynman-Kac formula. Theorem 3.1 can be generalized even further:

THEOREM 3.7 (Feynman-Kac Formula). *Under the standing assumptions, if $f(t,x)$ solves*

$$\begin{cases} A_t f(t,x) = r(t,x) f(t,x) & \text{for all } t \in [0,T] \\ f(T,x) = g(x) \end{cases}, \tag{7.10}$$

where $r(t,x)$ and $g(x)$ are some bounded functions, then

$$f(t,x) = \mathbb{E}\left(e^{-\int_t^T r(s,X_s)\,\mathrm{d}s} g(X_T) \middle| X_t = x \right).$$

Furthermore, $f(t,x)$ above is the unique solution to (7.10).

PROOF. The proof of uniqueness of the solution goes beyond the scope of this lecture and we do not prove it here. As in the previous cases, we want to show that the content of the expectation value is a martingale. Therefore, consider

$$M_\tau := e^{-\int_t^\tau r(s,X_s)\,\mathrm{d}s} f(\tau, X_\tau).$$

Defining $U_\tau = e^{-\int_t^\tau r(s,X_s)\,\mathrm{d}s}$, $Y_\tau = f(\tau, X_\tau)$ we apply Itô's formula to obtain

$$\mathrm{d}M_\tau = \mathrm{d}(U_\tau Y_\tau) = U_\tau \mathrm{d}(Y_\tau) + Y_\tau \mathrm{d}(U_\tau) + \mathrm{d}[U,Y]_\tau.$$

Recall from the chapter on the stochastic exponential that $U_\tau = \mathcal{E}(r(\tau, X_\tau))$ and that therefore $\mathrm{d}U_\tau = r(\tau, X_\tau) U_\tau \,\mathrm{d}\tau$. Furthermore, we recognize that U_τ has finite variation, so $\mathrm{d}[U,Y] = 0$. Combining these observations we obtain by Itô's formula for f, that

$$\begin{aligned} \mathrm{d}M_\tau = U_\tau \Bigg(& \left[\partial_\tau f(\tau, X_\tau) + \mu(\tau, X_\tau)\partial_x f(\tau, X_\tau) + \frac{1}{2}\sigma(\tau, X_\tau)^2 \partial_{xx}^2 f(\tau, X_\tau) \right] \mathrm{d}\tau \\ & + \sigma(\tau, X_\tau)\partial_x f(\tau, X_\tau)\,\mathrm{d}B_\tau - r(\tau, X_\tau) f(\tau, X_\tau) \mathrm{d}\tau \Bigg) \\ = {} & U_\tau \left((A_\tau - r(\tau, X_\tau)) f(\tau, X_\tau) + \sigma(\tau, X_\tau)\partial_x f(\tau, X_\tau)\,\mathrm{d}B_\tau \right). \end{aligned}$$

We immediately realize that the drift term in the above formula vanishes by assumption, and that the Itô integral term is a martingale by the standing assumptions on f, σ and μ. Consequently, the expected vylue of the martingale is constant and we have that

$$f(t,x)e^0 = M_t = \mathbb{E}[M_T | \mathcal{F}_t] = \mathbb{E}\left[e^{-\int_t^T r(s,X_s)\,\mathrm{d}s} g(X_T) \middle| X_t = x \right]$$

□

EXAMPLE 3.8 (Example 3.5 continued). *Let us consider the Black Scholes model i.e., $\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}B_t$ for $\sigma, \mu \in \mathbb{R}$. We consider the case where one can cash his/her option and obtain a risk-free interest that satisfies the* ODE

$$\mathrm{d}X_t = r X_t \,\mathrm{d}t,$$

for a positive constant $r \in \mathbb{R}$. Then, one needs to factor such possible, risk-free earning in the value $V(t, X_t)$ of the asset X_t (the underlying*), i.e., compare the expected value at future time T, $V(X_T) = V^*(X_T)$ with the* projected *risk-free value today:*

$$e^{r(T-t)} V(t, X_t) = \mathbb{E}\left[V^*(X_T) | X_t = x \right]$$

or, in other words,

$$V(t, X_t) = \mathbb{E}\left[e^{-r(T-t)} V(X_T) | X_t = x \right].$$

The above is an example of the expected value in Theorem 3.7, and therefore obeys the PDE

$$\begin{cases} \partial_t V(t,x) + \mu x \partial_x V(t,x) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} V(t,x) - r\,V(t,x) = 0 & \text{for all } t \in [0,T] \\ V(T,x) = V^*(x) \end{cases},$$

which is called the Black Scholes equation*.*

4. Time-homogeneous Diffusions

In this section we now consider a class of diffusion processes whose drift and diffusion coefficients do *not* depend explicitly on time:

DEFINITION 4.1. *If X_t solves*

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t \tag{7.11}$$

then X_t is a time-homogeneous *Itô diffusion process.*

Intuitively, the evolution of such processes does not depend on the time at which the process is started, *i.e.*, $\mathbb{P}[X_t \in A | X_0 = x_0] = \mathbb{P}[X_{t+s} \in A | X_s = x_0]$ for all $A \in \mathcal{B}(R)$, $s \in \mathcal{T}$ s.t. $t + s \in \mathcal{T}$. In other words, their evolution is invariant WRT translations in time, whence the name *time-homogeneous*.

DEFINITION 4.2. *Given a* SDE *with a unique solution we define the associated Markov semigroup P_t by*

$$(P_t\phi)(x) = \mathbb{E}_x \phi(X_t)$$

To see that this definition satisfies the semigroup property observe that the Markov property states that

$$(P_{t+s}\phi)(x) = \mathbb{E}_x\phi(X_{t+s}) = \mathbb{E}_x\mathbb{E}_{X_s}\phi(X_t) = \mathbb{E}_x(P_t\phi)(X_s) = (P_sP_t\phi)(x)\,.$$

Note that for the class of processes introduced above the infinitesimal generator is time-independent, *i.e.*, we have $A_t f = Af$. As a further consequence of the translation invariance (in time) of the SDE (7.11), the fact that the final condition of the backward Kolmogorov equation is at a specific time T is not relevant in this framework. This enables us to "store" the time-reversal in the function itself and look at the backward Kolmogorov equation as a forward equation as we explain below.

Let $f_-(x,t)$ be a bounded, $C^{2,1}$ function satisfying

$$\begin{aligned} \frac{\partial f_-}{\partial t} &= \mathcal{L} f_- \\ f_-(x,0) &= g(x) \end{aligned} \tag{7.12}$$

where L is the generator defined in (7.5). For simplicity we also assume that g is bounded and continuous. Then we have the analogous result to Theorem 3.1

THEOREM 4.3. *under $M_t = f_-(X_t, T-t)$ is a martingale for $t \in [0,T)$.*

PROOF. The proof is identical to the Brownian case. We start by applying Itô's formula

$$\begin{aligned} f_-(X_s, T-s) - f_-(X_0,T) = &\int_0^s \Big[-\frac{\partial f_-}{\partial t}(X_\gamma, T-\gamma) + \big(\mathcal{L}f_-\big)\big(X_\gamma, T-\gamma\big)\Big] d\gamma \\ &+ \int_0^s \frac{\partial f_-}{\partial x}\big(X_\gamma, T-\gamma\big)\,\mathrm{d}B_\gamma\ . \end{aligned}$$

As before the integrand of the first integral is identically zero because $\frac{\partial f_-}{\partial t} = \mathcal{L}f_-$. Hence only the stochastic integral is left on the right-hand side. □

And as before we have

Corollary 4.4. *In the above setting*

$$f_-(x,t) = \mathbb{E}[g\big(X_t\big)|X_0 = x]$$

The restriction to bounded and continuous g is not needed.

Proof of Cor. 4.4. Since $s \mapsto f_-(X_s, T-s) - f_-(X_0, T)$ is a martingale,

$$\mathbb{E}[f_-(X_T, 0)|X_0 = x] = \mathbb{E}[u(X_0, T)|X_0 = x]$$

because at $s = 0$ we see that $f_-(X_s, T-s) - u(X(0), T) = 0$. Since $\mathbb{E}[u(X_0, T)|X_0 = x] = u(x, T)$ and $\mathbb{E}[f_-(X_T, 0)|X_0 = x] = \mathbb{E}[g(X_T)|X_0 = x]$, the proof is complete. $\square$

For a more detailed discussion of Poisson and Dirichlet problems we refer to [**14**].

5. Stochastic Characteristics

To better understand Theorem 4.3 and Corollary 4.4, we begin by considering the deterministic case

$$\begin{aligned} \frac{\partial f_-}{\partial t} &= (b \cdot \nabla) f_- \\ f_-(x, 0) &= f(x) \end{aligned} \tag{7.13}$$

We want to make an analogy between the method of characteristics used to solve (7.13) and the results in Theorem 4.3 and Corollary 4.4. The method of characteristics is a method of solving (7.13) which in this simple setting amounts to finding a collection of curves ("characteristic curves") along which the solution is constant. Let us call these curves $x(t)$ were t is the parametrizing variable. Mathematically, we want $f_-(\xi(t), T-t)$ to be a constant independent of $t \in [0, T]$ for some fixed T. Hence the constant depends only on the choice of $\xi(t)$. We will look of $\xi(t) = (\xi_1(t), \cdots, \xi_d(t))$ which solve an ODE and thus we can parametrize the curves $\xi(t)$ by there initial condition $\xi(0) = x$. It may seem odd (and unneeded) to introduce the finial time T. This done so that $f_-(T, x) = f_-(T, \xi(0))$ and to keep the analogy close to what is traditionally done in SDEs. Differentiating $f_-(\xi(t), T-t)$ with respect to t, we see that maintaining constant amounts

$$\sum_{i=1}^d \frac{\partial f_-}{\partial x_i}(\xi(t), T-t)\frac{d\xi_i}{dt}(t) = \frac{\partial f_-}{\partial t}(\xi(t), T-t) = \sum_{i=1}^d b_i(\xi(t))\frac{\partial f_-}{\partial x_i}(\xi(t), T-t)$$

where the last equality follows from (7.13). We conclude that for this equality to hold in general we need

$$\frac{d\xi}{dt} = b(\xi(t)) \quad \text{and} \quad \xi(0) = x\,.$$

Since $f_-(\xi(t), T-t)$ is a constant we have

$$f_-(\xi(0), T) = f_-(\xi(T), 0) \Longrightarrow f_-(x, T) = f(\xi(T)) \tag{7.14}$$

which provides a solution to (7.13) to all points which can be reached by curves $\xi(T)$. Under mild assumptions this is all of $\mathbb{R}^d$.

Looking back at Theorem 4.3, we notice that differently from the *ode* case we did *not* find a SDE X_t which keeps $f_-(X_t, T-t)$ constant in the fully fledged sense. However, we have obtained something very close to it: We chose $t \mapsto f_-(X_t, T-t)$ to be a martingale, *i.e.*, a process that is constant *on average*! This is the content of Theorem 4.3 and Corollary 4.4 (putting the accent on the *expectation* part of the result), which mimicks the result of (7.14), only with the addition of expected values. Hence we might be provoked to make the following fanciful statement.

> *Stochastic differential equation are the method of characteristics for diffusions. Rather than follow a single characteristic back to the initial value to find the current value, we trace a infinite collection of stochastic curves each back to its own initial value which we then average weighting with the probability of the curve.*

6. A fundamental example: Brownian motion and the Heat Equation

We now consider the simple but fundamental case of standard Brownian motion.

Let us consider a compact subset $D \subset \mathbb{R}^2$ with a smooth boundary ∂D and a $f(x)$ defined on ∂D.

The Dirichlet problem: We are looking for a function $u(x)$ such that

$$\Delta u = \frac{\partial^2 u}{\partial^2 y_1} + \frac{\partial^2 u}{\partial^2 y_2} = 0 \text{ for } y = (y_1, y_2) \text{ inside } D.$$
$$\lim_{y \to x} u(y) = f(x) \text{ for all } x \in \partial D.$$

Let $B(t,\omega) = (B_1(t,\omega), B_2(t,\omega))$ be a two dimensional Brownian motion. Define the stopping time

$$\tau = \inf\{t > 0 : B(t) \notin D\}$$

Let $\mathbb{E}_y$ be the expectation with respect to the Wiener measure for a Brownian motion starting from y at time $t = 0$. Let us define $\phi(x) = \mathbb{E}_y f(B(\tau))$. We are going to show that $\phi(x)$ solves the Dirichlet problem.

Lemma 6.1. *With probability* 1, $\tau < \infty$. *In fact,* $\mathbb{E}\tau^r < \infty$ *for all* $r > 0$.

Proof.

$$\mathbb{P}\{\tau \geqslant n\} \leqslant \mathbb{P}\{|B(1) - B(0)| \leqslant \text{diam}D, |B(2) - B(1)| \leqslant \text{diam}D, \dots, |B(n) - B(n-1)| \leqslant \text{diam}D\}$$
$$\leqslant \prod_{k=1}^{n} \mathbb{P}\{||B(k) - B(k-1)| \leqslant \text{diam}D\} = \alpha^n \text{ where } \alpha \in (0,1)$$

Hence $\sum_{n=1}^{\infty} \mathbb{P}\{\tau \geqslant n\} < \infty$ and the Borel-Cantelli lemma says that τ is almost surely finite. Now lets look at the moments.

$$\mathbb{E}\tau^r = \int x^r \mathbb{P}\{\tau \in dx\} \leqslant \sum_{n=1}^{\infty} n^r \mathbb{P}\{\tau \in (n-1, n]\} \leqslant \sum_{n=1}^{\infty} n^r \mathbb{P}\{\tau \geqslant n-1\} \leqslant \sum_{n=1}^{\infty} n^r \alpha^n < \infty$$

□

Lets fix a point y in the interior of D. Lets put a small circle of radius ρ around y so that the circle in contained completely in D. Let $\tau_{\rho,y}$ be the first moment of time $B(t)$ hits the circle of radius ρ centered at y.

Because the law of Brownian motion is invariant under rotations, we see that $B(\tau_{\rho,y})$ is distributed uniformly on the circle centered at y. (Lets call this circle $S_\rho(y)$.)

Theorem 6.2. $\phi(x)$ *solves the Laplace equation.*

Proof. i) We start by proving the mean value property. To do so we invoke the Strong Markov property of $B(t)$. Let $\tau_S = inf\{t : B(t) \in S_\rho(y)\}$ and $z_\vartheta = (\rho\cos\vartheta, \rho\sin\vartheta)$ be the point on $S_\rho(y)$ at angle ϑ. We notice that any path from y to the boundary of D must pass through $S_\rho(y)$. Thus we can think of $\phi(y)$ as the weighted average of $\mathbb{E}\{f(B(\tau))\big|B(\tau_S) = z_\vartheta$ where ϑ moves use around the circle $S_\rho(y)$. Each entry in this average is weighted by the chance of hitting that point on the sphere starting from y. Since this chance is uniform (all points are equally likely), we simply get the factor of $\frac{1}{2\pi}$ to normalize things to be a probability measure.

$$\phi(y) = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta \mathbb{E}\{f(B(\tau))\big|B(\tau_S) = z_\vartheta\} = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta \phi(z_\vartheta) \tag{7.15}$$

ii) $\phi(x)$ is infinitely differentiable. This can be easily shown but let us just assume it since we are doing this exercise in explicit calculation to improve our understanding, not to prove every detail of the theorems.

iii) Now we see that ϕ satisfies $\Delta\phi = \frac{\partial^2\phi}{\partial y_1^2} + \frac{\partial^2\phi}{\partial y_2^2} = 0$. We expand about a point y in the interior of D.

$$\begin{aligned}\phi(z) =&\phi(y) + \frac{\partial\phi}{\partial y_1}(z_1 - y_1) + \frac{\partial\phi}{\partial y_2}(z_2 - y_2) \\ &+ \frac{1}{2}\left[\frac{\partial^2\phi}{\partial y_1^2}(z_1 - y_1)^2 + \frac{\partial^2\phi}{\partial y_2^2}(z_2 - y_2)^2 + \frac{\partial^2\phi}{\partial y_1 \partial y_2}(z_1 - y_1)(z_2 - y_2)\right] + O(|z - y|^3)\end{aligned}$$

Now we integrate this over a circle $S_\rho(y)$ centered at y of radius ρ. We take ρ to by sufficiently small so that the entire disk in the domain D. By direct calculation we have

$$\int_{S_\rho(y)}(z_1 - y_1)dz = 0, \int_{S_\rho(y)}(z_2 - y_2)dz = 0, \int_{S_\rho(y)}(z_1 - y_1)(z_2 - y_2)dz = 0$$

and

$$\int_{S_\rho(y)}(z_1 - y_1)^2dz = (\text{const})\rho^2, \quad \int_{S_\rho(y)}(z_2 - y_2)^2dz = (\text{const})\rho^2 \ .$$

Since by the mean value property,

$$\phi(y) = (\text{const})\int_{S_\rho(y)}\phi(z)dz$$

we see that

$$0 = (\text{const})\rho^2\left(\frac{\partial^2\phi}{\partial y_1^2} + \frac{\partial^2\phi}{\partial y_2^2}\right) + O(\rho^3) \ .$$

And thus,

$$\Delta\phi = \frac{\partial^2\phi}{\partial y_1^2} + \frac{\partial^2\phi}{\partial y_2^2} = 0$$

□

CHAPTER 8

Martingales and Localization

This chapter is dedicated to a more in-depth study of martingales and their properties. Some of the results exposed here are fairly general, and their proof in full generality required tools that are more advanced than the ones we have at our disposal. For this reason, some of the proofs will be given in a simplified setting/under stronger assumptions together with a reference for the more general result.

1. Martingales & Co.

We recall the definition of a martingale given at the beginning of the course, extending it slightly.

Definition 1.1. *$\{X_t\}$ is a* Martingale with respect to a filtration $\mathcal{F}_t$ *($\mathcal{F}_t$-martingale for short) if for all $t > s$ we have*

i) *X_t is $\mathcal{F}_t$-measurable,*

ii) $\mathbb{E}[|X_t|] < \infty$ *,*

iii) $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ *a.s..*

*Similarly, X_t is a $\mathcal{F}_t$-*supermartingale [$\mathcal{F}_t$-submartingale] *is it satisfies conditions i) and ii) above, and*

$$\mathbb{E}[X_t|\mathcal{F}_s] \leqslant X_s\,, \qquad [\mathbb{E}[X_t|\mathcal{F}_s] \geqslant X_s] \quad a.s..$$

When the filtration is clear from the context we simply say that a process is a [super/sub-]martingale.

Super- and Submartingales extend the idea of a process that is constant in expectation to processes that are, respectively, nonincreasing and nondecreasing in expectation. It is clear that a martingale is both a supermartingale and a submartingale, while a supermartingale that is also a submartingale is a martingale.

Proposition 1.2. *A supermartingale [submartingale] M_t is a martingale on $[0,T]$ if and only if $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.*

Proof. The "only if" direction follows by definition: if M_t is a martingale then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$ and it is both a super- and a submartingale. For the "if" assume that M_t is a supermartingale and $\mathbb{E}[M_T] = \mathbb{E}[M_0]$. Assume by contradiction that it is not a martingale, *i.e.*, that there is a set $W \subseteq \Omega$ of positive probability such that $\mathbb{E}[M_t|\mathcal{F}_s] < M_s$ for all $\omega \in W$. Then by the supermartingale property of M_t we have that

$$\mathbb{E}[M_T] \leqslant \mathbb{E}[M_t] = \mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_s]] < \mathbb{E}[M_s] \leqslant \mathbb{E}[M_0]\ ,$$

which contradicts the assumption. □

Remark 1.3. *By Jensen's inequality on conditional expectations we have for any convex function $g : \mathbb{R} \to \mathbb{R}$, a martingale M_t satisfies*

$$\mathbb{E}[g(M_t)|\mathcal{F}_t] \geqslant g(\mathbb{E}[M_t|\mathcal{F}_s]) = g(M_s)\,,$$

so application of a convex [concave] map to a martingale makes it a submartingale [supermartingale].

Recall that a random variable X is [square-]integrable if $\mathbb{E}[|X|] < \infty$ [$\mathbb{E}\left[X^2\right] < \infty$]. The condition of simple integrability of a random variable X can be equivalently stated as the condition

$$\lim_{n\to\infty} \mathbb{E}\left[|X|\mathbb{1}_{|X|>n}\right] = 0\,. \tag{8.1}$$

Indeed, on one hand as $\lim_{n\to\infty} |X|\mathbb{1}_{|X|>n} = 0$ a.s. and $\mathbb{E}\left[|X|\mathbb{1}_{|X|>n}\right] \leqslant \mathbb{E}[|X|] < \infty$ we have by the dominated convergence theorem[1] that $\lim_{n\to\infty} \mathbb{E}\left[|X|\mathbb{1}_{|X|>n}\right] = \mathbb{E}[0] = 0$. On the other hand, we have that

$$\mathbb{E}[|X|] = \mathbb{E}\left[|X|\mathbb{1}_{|X|\leqslant n}\right] + \mathbb{E}\left[|X|\mathbb{1}_{|X|>n}\right] \leqslant n + \mathbb{E}\left[|X|\mathbb{1}_{|X|>n}\right]\,, \tag{8.2}$$

and by using that both summands on the right hand side are bounded (the first by definition and the second by assumption) we obtain that $\mathbb{E}[|X|] < \infty$.

We now generalize the definitions above to stochastic processes.

Definition 1.4. *A stochastic process X_t on $\mathcal{T} = [0, T]$ (where possibly $T = \infty$) is*

i) integrable *if* $\sup_{t\in\mathcal{T}} \mathbb{E}[|X_t|] < \infty$,

ii) square integrable *if* $\sup_{t\in\mathcal{T}} \mathbb{E}\left[X_t^2\right] < \infty$ *(i.e., the second moments are uniformly bounded)*,

iii) uniformly integrable *if* $\lim_{n\to\infty} \sup_{t\in\mathcal{T}} \mathbb{E}\left[|X_t|\mathbb{1}_{|X_t|>n}\right] = 0$

The introduction of (8.1) allows to separate the concepts of simple and uniform integrability for stochastic processes as in the latter definition the limit is taken *after* the supremum. As one would expect, uniform integrability is stronger than simple integrability: similarly to (8.2) one has

$$\sup_{t\in\mathcal{T}} \mathbb{E}[|X_t|] = \sup_{t\in\mathcal{T}} \mathbb{E}\left[|X_t|\mathbb{1}_{|X_t|\leqslant n}\right] + \sup_{t\in\mathcal{T}} \mathbb{E}\left[|X_t|\mathbb{1}_{|X_t|>n}\right] \leqslant n + \sup_{t\in\mathcal{T}} \mathbb{E}\left[|X_t|\mathbb{1}_{|X_t|>n}\right] < \infty\,.$$

For the converse result we need stronger assumptions. We give below examples of such results:

Proposition 1.5. *A stochastic process $\{X_t\}$ is uniformly integrable if, either*

i) *It is dominated by a random variable Y defined on the same probability space, i.e., $X_t(\omega) \leqslant Y(\omega)$ such that $\mathbb{E}[|Y|] < \infty$,*

ii) *There exists some positive function $G(x)$ on $(0, \infty)$ with $\lim_{x\to\infty} G(x)/x = \infty$ such that*

$$\sup_{t\in\mathcal{T}} \mathbb{E}[G(|X_t|)] < \infty\,.$$

Proof. We only prove the first result, for which we have that

$$\lim_{n\to\infty} \sup_{t\in\mathcal{T}} \mathbb{E}\left[|X_t|\mathbb{1}_{|X_t|>n}\right] \leqslant \lim_{n\to\infty} \mathbb{E}\left[|Y|\mathbb{1}_{|Y|>n}\right] < \infty\,.$$

For the proof of the second result we refer *e.g.*, to [**16**]. □

In ii) of the above theorem we see that we need something *sligthly* better than simple integrability to have uniform integrability. In particular we see that $G(x) = x^{1+\varepsilon}$ for any $\varepsilon > 0$ satisfies condition ii) of the above theorem. In particular, all square integrable martingales are uniformly integrable.

Theorem 1.6. *Let Y be an integrable random variable on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$, then*

$$M_t := \mathbb{E}[Y|\mathcal{F}_t] \tag{8.3}$$

is a uniformly integrable martingale.

Proof. We refer to Klebaner [**9**, Proof of Thm. 7.9]. □

[1] See Theorem 0.2 in the appendix for a reminder of this theorem

We define a martingale such as the one in (8.3) as *closed* by the random variable Y. In particular, for any finite time interval $[0,T]$ by definition every martingale is closed by its value at T since $\mathbb{E}[M_T|\mathcal{F}_t] = M_t$ and we have the following corollary.

COROLLARY 1.7. *Any martingale M_t on a finite time interval is uniformly integrable.*

The above results can be extended to infinite time intervals.

THEOREM 1.8 (Martingale convergence theorem). *Let M_t on $\mathcal{T} = [0,\infty)$ be an integrable [sub/super]-martingale. Then there exists an almost sure (i.e., pointwise) limit $\lim_{t\to\infty} M_t = Y$, and Y is an integrable random variable.*

The above theorem does *not* establish a correspondence between the random variables in terms of expected values. In particular, we may have cases where the theorem above applies but we have $\lim_{t\to\infty}\mathbb{E}[M_t] \neq \mathbb{E}[Y]$:

EXAMPLE 1.9. *Consider the martingale $M_t = \exp[B_t - t/2]$. Because it is positive, we have that*

$$\sup_{t\in\mathcal{T}} \mathbb{E}[|M_t|] = \sup_{t\in\mathcal{T}} \mathbb{E}[M_t] = \mathbb{E}[M_0] = 1 < \infty\,,$$

so it converges almost surely to a random variable Y by Theorem 1.8. However, we see that by the law of large numbers for Brownian motion $B_t/t \to 0$ a.s. and therefore

$$\mathbb{E}[Y] = \mathbb{E}\left[\lim_{t\to\infty} M_t\right] = \mathbb{E}\left[\lim_{t\to\infty} e^{t(B_t/t-1/2)}\right] = 0\,,$$

which differs from $\lim_{t\to\infty}\mathbb{E}[M_t] = 1$.

The above observation neans in particular that the conditions of Theorem 1.8 do not guarantee convergence in the L^1 norm. Under the stronger condition of *uniform* integrability of the process X_t one obtains the same result with convergence in L^1 norm and consequently the closedness of the martingale:

THEOREM 1.10. *Let M_t be a uniformly integrable martingale on $\mathcal{T} = [0,\infty)$, then it converges as $t\to\infty$ in L^1 and a.s. to a random variable Y. Conversely, if M_t converges in L^1 to an integrable random variable Y then it is square integrable and converges almost surely. In both cases M_t is closed by Y.*

2. Optional stopping

After studying martingales *per se*, we consider their relation with stopping times. In particular, we will see that martingales behave nicely with respect to stopping times. To be more explicit, given a stochastic process X_t and recalling the definition Def. 7.14 of a stopping time τ, we denote by $\tau\wedge t = \min(\tau,t)$ and define the *stopped process*

$$X_t^\tau := X_{\tau\wedge t} = \begin{cases} X_t & \text{if } t<\tau \\ X_\tau & \text{else} \end{cases}\,.$$

The following theorem gives an example of the nice relationship between martingales and stopping times: it says that the martingale property is maintained by a process when such process is stopped.

THEOREM 2.1. *For a $\mathcal{F}_t$-martingale M_t and any stopping time τ, the process $M_{\tau\wedge t}$ is a $\mathcal{F}_t$-martingale (and therefore a $\mathcal{F}_\tau \wedge t$-martingale), so*

$$\mathbb{E}[M_{\tau\wedge t}] = \mathbb{E}[M_0] \quad \textit{for all } t>0\,. \tag{8.4}$$

Martingales are often thought of as *fair games* because of their property of conserving their expected value: It is impossible, on average, to make positive gains by playing such game. Under this interpretation, Theorem 2.1 states that even if a player is given the possibility of quitting the game use any betting strategy, he/she will not be able to make net gains at time t provided that his/her strategy only depends on past information (cfr. Def. 7.14 of stopping time). However, the above property is lost if the player is patient enough, as the following example shows:

EXAMPLE 2.2. *Let B_t be a Standard Brownian motion (a martingale, hence an example of a "fair game": you can think of it as a continuous version of betting one dollar on every coin flip) and define $\tau_1 := \inf\{t \;:\; B_t \geqslant 1\}$ (the strategy of stopping as soon as you have a net gain of 1\$). Then by definition we have that $B_\tau = 1 \neq 0 = \mathbb{E}[B_0]$.*

A similar situation to the one described above holds when considering the "martingale" betting strategy of doubling your bet every time you loose a coin flip. This strategy leads to an almost sure net win of 1$ if one is patient enough (and has enough money to bet). As the examples above shows, stopped martingales may lose the property of conserving the expected value in the limit $t \to \infty$. The following theorem gives sufficient conditions for the martingale property to hold in this limit, *i.e.*, for the expected value of a game to be conserved at a stopping time τ:

THEOREM 2.3 (Optional stopping theorem). *Let M_t be a martingale, τ a stopping time, then we have $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$ if either of the following conditions holds:*

- *The stopping time τ is bounded a.s, i.e., $\exists K < \infty$: $\tau \leqslant K$,*
- *The martingale M_t is uniformly integrable ,*
- *The stopping time is finite a.s. (i.e., $\mathbb{P}[\tau = \infty] = 0$), M_t is integrable and*

$$\lim_{t\to\infty} \mathbb{E}[M_t \mathbb{1}_{\tau > t}] = 0\,.$$

PROOF. □

Under the gaming interpretation of above, we see that a game is "fair", *i.e.*, it is impossible to make net gains, on average, using only past information, if any of the conditions i)-iii) hold. In particular, in the case of coin-flip games (or casino games) we see that a winning strategy does not exist as condition ii) holds: there is only a finite amount of money in the world, so the martingale is uniformly bounded, and in particular uniformly integrable. A simplified example of such a situation is given next:

EXAMPLE 2.4. *Let B_t be a Standard Brownian motion on on the interval $a < 0 < b$ and define the stopping time $\tau = \tau_{ab} = \inf\{t \in [0,\infty) \;:\; B_t \notin (a,b)\}$. The stopped process $B_\tau \wedge t$ is uniformly bounded and in particular uniformly integrable. Hence, by Theorem 2.3 we have that $\mathbb{E}[B_\tau] = \mathbb{E}[B_0] = 0$. However, we also have that $B_\tau = b$ with probability p and $B_\tau = a$ with probability $1-p$, therefore*

$$0 = \mathbb{E}[B_\tau] = a \cdot (1-p) + b \cdot p \quad \Rightarrow \quad \mathbb{P}[B_\tau = b] = p = \frac{-a}{b-a}\,,$$

which we have concluded based on considerations based on the martingale properties of B_t and therefore extends to any martingale for which τ_{ab} is finite a.s..

We conclude the chapter by presenting the converse of Theorem 2.3:

PROPOSITION 2.5. *Let X_t be a stochastic process such that for any stopping time τ, X_τ is integrable and $\mathbb{E}[X_0] = \mathbb{E}[X_\tau]$. Then X_t is a martingale.*

PROOF. We refer to Klebaner [**9**, Proof of Thm. 7.17]. □

3. Localization

This section is devoted to the use of stopping times for the study of the properties of stochastic processes. As we have seen, the stopped process may have some properties that the original process did not have (*e.g.*, uniform integrability on $[0, \infty)$ in Example 2.4). One can generalize such situation to a *sequence* of stopping times, such as the following example:

EXAMPLE 3.1. *Consider, similarly to Example 2.4, a Standard Brownian motion B_t on the interval $(-n, n)$ for $n \in \mathbb{N}$. Then we can define the stopping times $\tau_n := \inf\{t \,:\, B_t \notin (-n, n)\}$. For each $n > 0$, the process is uniformly integrable.*

In the above example, by taking the limit $n \to \infty$ one would approach the original setting of unbounded Brownian motion by approximating it with uniformly bounded stopped processes. This prices can be extremely useful to obtain stronger results as the ones obtained previously in the course, as we will see later in this section, and justifies the following definition:

DEFINITION 3.2. *A property of a stochastic process X_t is said to hold* locally *if there exists a sequence $\{\tau_n\}$ of stopping times with the property $\lim_{n\to\infty} \tau_n(\omega) = \infty$ a.s. such that the stopped process $X_{\tau_n \wedge t}$ has such property. In this case, the sequence $\{\tau_n\}$ is called the localizing sequence.*

A particularly useful example is the one of the martingale property:

DEFINITION 3.3. *An adapted process M_t is a local martingale if there exists a sequence of stopping times $\{\tau_n\}$ such that $\lim_{n\to\infty} \tau_n(\omega) = \infty$ a.s. and the stopped process $M_{\tau_n \wedge t}$ is a martingale for all n.*

It is clear that if a property holds in the original sense, then it also holds locally: one just has to take $\tau_n = n > t$. On the contrary a local martingale is in general not a martingale:

EXAMPLE 3.4. *Consider the Itô integral $\int_0^t \exp[B_s^2] \,\mathrm{d}B_s$, for $t < 1/4$ and define $\tau_n := \inf\{t > 0 \,:\, \exp[B_s^2] = n\}$. The process $M_{\tau_n \wedge t}$ is a martingale, since we can write it as*

$$M_{\tau_n \wedge t} = U_t = \int_0^t \exp[B_s^2] \mathbb{1}_{\exp[B_s^2] \leqslant n} \,\mathrm{d}B_s$$

is square integrable by Itô isometry. However, we have that

$$\mathbb{E}\left[\exp[2B_t^2]\right] = \frac{1}{2\pi t} \int_{-\infty}^{\infty} e^{2x^2} e^{-x^2/(2t)} \,\mathrm{d}x$$

which diverges for $t > 1/4$, implying that M_t is not integrable.

We now list some results that, besides allowing to practice the use of localization methods, give sufficient conditions for a local martingale to be a martingale.

PROPOSITION 3.5. *Let M_t be a local martingale such that $|M_t| <\leqslant Y$ for an integrable random variable Y, then M_t is a uniformly integrable martingale.*

PROOF. Let τ_n be a localizing sequence, then for any n and $s < t$ we have

$$\mathbb{E}\left[M_{t \wedge \tau_n} | \mathcal{F}_s\right] = M_{s \wedge \tau_n}\,.$$

Because $\tau_n \uparrow \infty$ a.s. we have the pointwise convergence $\lim_{n\to\infty} X_{s\wedge\tau_n} = X_s$. Furthermore by our assumptions M_t is integrable, and we can apply Dominated Convergence Theorem[2] to obtain that

$$\mathbb{E}\left[M_t | \mathcal{F}_s\right] = \mathbb{E}\left[\lim_{n\to\infty} X_{t\wedge\tau_n} | \mathcal{F}_s\right] = \lim_{n\to\infty} \mathbb{E}\left[X_{t\wedge\tau_n} | \mathcal{F}_s\right] = \lim_{n\to\infty} X_{s\wedge\tau_n} = M_s\,,$$

showing that M_t is a martingale. By Proposition 1.5 we establish uniform integrability of M_t. □

[2]a version of this theorem is presented in the appendix

PROPOSITION 3.6. *A non-negative local martingale M_t, for $t \in [0,T]$ is a supermartingale.*

PROOF. Let $\{\tau_n\}$ be the localizing sequence of M_t. Then for any t we have that $\lim_{n\to\infty} \tau_n \wedge t = t$ a.s and therefore that $\lim_{n\to\infty} M_{\tau_n \wedge t} = M_t$. Consequently, by Fatou's lemma on conditional expectations we have

$$\mathbb{E}\left[M_t|\mathcal{F}_s\right] = \mathbb{E}\left[\liminf_{n\to\infty} M_{\tau_n\wedge t}|\mathcal{F}_s\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[M_{\tau_n\wedge t}|\mathcal{F}_s\right] = \liminf_{n\to\infty} M_{\tau_n\wedge s} = M_s \quad \text{a.s.}\,,$$

where in the second equality we have used that the limit exists. In particular, we have that $\mathbb{E}\left[M_t\right] \leqslant \mathbb{E}\left[M_0\right] < \infty$. ☐

COROLLARY 3.7. *A non-negative local martingale M_t on $\mathcal{T} = [0,T]$ for $T < \infty$ is a martingale if and only if $\mathbb{E}\left[M_T\right] = M_0$*

PROOF. This is a direct result of Proposition 1.2 and Proposition 3.6. ☐

REMARK 3.8. *As explained in* [**9**] *there exists a necessary and sufficient condition for a local martingale to be a martingale: for the local martingale to be of "Dirichlet Class", i.e., such that such that the collection of random variables*

$$\mathbf{X} = \{X_\tau \;:\; \tau \text{ is a finite stopping time}\}$$

is uniformly integrable, i.e., $\sup_{X\in\mathbf{X}} \lim_{n\to\infty} \mathbb{E}\left[|X|\mathbb{1}_{|X|>n}\right] = 0$.

We now give some slightly more advanced examples of the use of localization procedure. We begin by revisiting the problem of proving moment bounds for Itô integrals.

Moment Bounds for Itô Integrals. We let $I_t = \int_0^t \sigma_s dB_s$. We want to prove the moment bounds

$$\mathbb{E}|I_t|^{2p} \leqslant (2p-1)(2p-3)\cdots 3\cdot 1\cdot M^{2p}t^p\,,$$

under the assumption that $|\sigma_s| \leqslant M$ a.s.

The case $p = 1$ follows from the Itô isometry. Therefore, we now proceed to prove the induction step. Let us assume the inequality for $p-1$ and use it to prove the inequality for p. For any $N > 0$, we define

$$\tau_N = \inf\{t \geqslant 0 : \int_0^t |I_s|^{4p-2}\sigma_s^2 ds \geqslant N\}$$

Applying Itô formula to $x \mapsto |x|^{2p}$ and evaluating at the time $t \wedge \tau_N$ produces

$$|I_{t\wedge\tau_N}|^{2p} = p(2p-1)\int_0^{t\wedge\tau_N} |I_s|^{2(p-1)}\sigma_s^2\,\mathrm{d}s + 2p\int_0^{t\wedge\tau_N} |I_s|^{2p-1}\sigma_s dB_s = (I) + (II)$$

now by the induction hypothesis

$$\begin{aligned}\mathbb{E}(I) &\leqslant p(2p-1)\int_0^t \mathbb{E}|I_s|^{2(p-1)}\sigma_s^2\,\mathrm{d}s \leqslant (2p-1)(2p-3)\cdots 3\cdot 1\cdot pM^{2p}\int_0^t s^{p-1}\,\mathrm{d}s\\ &\leqslant (2p-1)(2p-3)\cdots 3\cdot 1\cdot M^{2p}(t\wedge\tau_N)^p \leqslant (2p-1)(2p-3)\cdots 3\cdot 1\cdot M^{2p}t^p\end{aligned}$$

If we define

$$U_t = \int_0^t |I_s|^{2p-1}\sigma_s \mathbf{1}_{s\leqslant\tau_N} dB_s$$

then U_t is a martingale since

$$\int_0^t |I_s|^{4p-2}|\sigma_s|^2\mathbf{1}_{s\leqslant\tau_N}\,\mathrm{d}s = \int_0^{t\wedge\tau_N} |I_s|^{4p-2}|\sigma_s|^2\,\mathrm{d}s \leqslant N\,.$$

Since $t \wedge \tau_N$ is a bounded stopping time, the optional stopping lemma says that $\mathbb{E}U_{t\wedge\tau_N} = 0$. However as noted above

$$\mathbb{E}(II) = \mathbb{E}U_{t\wedge\tau_N}$$

so one obtains

$$\mathbb{E}|I_{t\wedge\tau_N}|^{2p} \leqslant (2p-1)(2p-3)\cdots 3\cdot 1\cdot M^{2p}t^p\,.$$

Since $|I_s|$ is almost surely finite, we know that τ_N is finite with probability one. Hence $|I_{t\wedge\tau_N}|^{2p} \to |I_t|^{2p}$ almost sure. Then by Fatou's lemma we have

$$\mathbb{E}|I_t|^{2p} \leqslant \lim_{N\to\infty} \mathbb{E}|I_{t\wedge\tau_N}|^{2p} \leqslant (2p-1)(2p-3)\cdots 3\cdot 1\cdot M^{2p}t^p \tag{8.5}$$

SDEs with Superlinear Coefficients. Let $b\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma^{(i)} : \mathbb{R}^d \to \mathbb{R}^d$ be such that for any $R > 0$ there exists a C such that

$$|b(x) - b(y)| + \sum_{i=1}^m |\sigma^{(i)}(x) - \sigma^{(i)}(y)| \leqslant C|x-y|$$
$$|b(x)| + |\sigma(x)| \leqslant C$$

for any $x, y \in \mathcal{B}_0(R)$, where $\mathcal{B}_0(r) := \{x \in \mathbb{R}^d \;:\; \|x\|_2 < R\}$.

Consider the SDE

$$dX_t = b(X_t)\,\mathrm{d}t + \sum_{i=1}^m \sigma^{(i)}(X_t)\,\mathrm{d}B_t^{(i)} \tag{8.6}$$

For any R let b_R and $\sigma_R^{(i)}$ be are globally bounded and globally Lipchitz functions in $\mathbb{R}^d$ such that $b_R(x) = b(x)$ and $\sigma_R(x) = \sigma(x)$ in $\mathcal{B}_0(R)$.

Since b_R and σ_R satisfy the existence and uniqueness assumptions of Chapter 6.3, there exists a solution $X_t^{(R)}$ to the equation

$$dX_t^{(R)} = b_R(X_t^{(R)})\,\mathrm{d}t + \sum_{i=1}^m \sigma_R^{(i)}(X_t^{(R)})\,\mathrm{d}B_t^{(i)} \tag{8.7}$$

For any $N > 0$ and $R > 0$ we define the stopping time

$$\tau_R := \inf\{t \geqslant 0 : |X_t^{(R)}| > R\}$$

THEOREM 3.9. *If*

$$\mathbb{P}\left[\lim_{R\to\infty} \tau_R = \infty\right] = 1$$

there there exists a unique strong solution to (8.6)*.*

PROOF. Fix a $T > 0$. For $R \in \mathbf{N}$ let $\Omega_R = \{\tau_R < T < \tau_{R+1}\}$. By the assumption

$$\mathbb{P}\left[\bigcup_{R=1}^{\infty} \Omega_R\right] = 1\,.$$

Also notice that the Ω_R are disjoint for $R = 1, 2, \ldots$ and we can define the process

$$X_t(\omega) = X_t^{(R)}(\omega) \text{ for } t \in [0,T] \text{ if } \omega \in \Omega_R\,.$$

Since $\sup|X_t^{(R)}(\omega)| < R$, we know that $b(X_t^{(R)}(\omega)) = b^{(R)}(X_t^{(R)}(\omega))$ and $\sigma(X_t^{(R)}(\omega)) = \sigma^{(R)}(X_t^{(R)}(\omega))$ for all $t \in [0,T]$. Hence X_t as defined solves the original equation. Uniqueness comes from the fact that solutions to (8.7) are unique. $\square$

4. Quadratic variation for martingales

Recall the definition of quadratic variation of a stochastic process:

DEFINITION 4.1. *The* quadratic variation *of an adapted stochastic process X_t is defined as*

$$[X]_t := \lim_{N\to\infty}^{p} \sum_{j=0}^{j^N} \left(X_{t_{j+1}^N} - X_{t_j^N} \right)^2$$

where $\lim^p$ *denotes a limit in probability and* $\{t_j^N\}$ *is a set partitioning the interval* $[0,t]$ *defined by*

$$\Gamma^N := \{\{t_j^N\} \ : \ 0 = t_0^N < t_1^N < \cdots < t_{j^N}^N = t\} \tag{8.8}$$

with $|\Gamma^N| := \sup_j |t_{j+1}^N - t_j^N| \to 0$ *as* $N \to \infty$.

The process defined above is a sum of positive contributions and is therefore nondecreasing in t a.s..

Now let M_t be a [local] martingale. In light of Remark 1.3 we know that M_t^2 is a [local] submartingale. Hence, we would like to know if we can transform M_t^2 back to a martingale, for example by subtracting a "compensation process" removing the nondecreasing part of the squared process. It turns out that such process exists and is precisely the quadratic variation process. The intuition behind this result comes from the following computation: assume that $s < t$, then we have

$$\mathbb{E}\left[M_t M_s\right] = \mathbb{E}\left[M_s \mathbb{E}\left[M_t|\mathcal{F}_s\right]\right] = \mathbb{E}\left[M_s^2\right]$$

where in the second equality we have used the martingale property. As a consequence of this we can write

$$\mathbb{E}\left[(M_t - M_s)^2\right] = \mathbb{E}\left[M_t^2\right] - 2\mathbb{E}\left[M_t M_s\right] + \mathbb{E}\left[M_s^2\right] = \mathbb{E}\left[M_t^2\right] - \mathbb{E}\left[M_s^2\right] . \tag{8.9}$$

In particular this implies that the summands in the definition of quadratic variation can be expressed, on expectation, as differences of expectation values that cancel telescopically, leading to (part of) the following theorem.

THEOREM 4.2. *This theorem can be stated in the martingale and local martingale version:*

i) *Let M_t be a square-integrable martingale, then the quadratic variation process $[M]_t$ exists and $M_t^2 - [M]_t$ is a martingale.*
ii) *Let M_t be a local martingale, then the quadratic variation process $[M]_t$ exists and $M_t^2 - [M]_t$ is a local martingale.*

PROOF. We only prove point i) of the theorem above. Point ii) follows for locally square integrable martingales by localization, *i.e.*, by substituting $t \to \tau_n \wedge t$ where τ_n is the localizing sequence. Repeating the calculation leading to (8.9) with conditional expectations we obtain

$$\mathbb{E}\left[M_t^2 - M_s^2|\mathcal{F}_s\right] = \mathbb{E}\left[\sum_{j=1}^{j^N} (M_{t_j^N} - M_{t_{j-1}^N})^2 |\mathcal{F}_{t_{j-1}^N}\right] .$$

Now, taking the limit in probability of the right hand side (we do not prove that such limit exists here, but we refer to [**16**]) and rearranging we obtain that $\mathbb{E}\left[M_t^2 - [M]_t|\mathcal{F}_s\right] = M_s^2 - [M]_s$ as desired. □

We conclude this section by proving a surprising result about martingales with finite first variation.

LEMMA 4.3. *Let M_t be a continuous local martingale with finite first variation. Then M_t is almost surely constant.*

The intuition behind the above result is quite simple: considering a continuum time interval, constraining a continuous martingale on behaving "nicely" in order to have finite first variation (for example monotonically or in a differentiable way) the martingale would somehow have to be "consistent with its trend at t_-" (except of course in a set of measure 0) and could therefore not respect the constant conditional expectation property. In other words, martingales with finite first variation are too "stiff" to be different from the identity function.

REMARK 4.4. *Note that continuity is a key requirement in the above result: jump processes (constant between jumps, discontinuous when jumps occur) give an example of martingales that are not constant but that have finite first variation.*

PROOF OF LEMMA 4.3. We assume for this proof that M_t is a [locally] bounded martingale. We will eventually show that the variance of M_t is zero and hence M_t is constant. Picking some partition of time $0 = t_0 < t_1 < \cdots < t_k = t$, recalling (8.9) we consider the variance at time t

$$\begin{aligned}\mathbb{E}M_t^2 &= \mathbb{E}\sum\left(M_{t_n}^2 - M_{t_{n-1}}^2\right) = \mathbb{E}\sum(M_{t_n} - M_{t_{n-1}})^2 \\ &\leqslant \mathbb{E}\sup_{t_n}|M_{t_n} - M_{t_{n-1}}|\sum|M_{t_n} - M_{t_{n-1}}|\end{aligned}$$

Since the first variation $V(t) = \lim_{\Delta T\to 0}\sum|M_{t_n} - M_{t_{n-1}}|$ was assumed to be finite we obtain

$$\mathbb{E}M_t^2 \leqslant (\text{const})\mathbb{E}\lim_{\Delta T\to 0}\sup_{t_n}|M_{t_n} - M_{t_{n-1}}|$$

this limit is zero because M was assumed to be continuous.

Hence the variance of M_t is zero and thus M_t is constant almost surely. Thus M_t is constant for any countable collection of times. Use the rational numbers and then continuity to conclude that it is constant and the same constant for all times. □

5. Lévy-Doob characterization of Brownian motion

In the beginning of this course we have given several equivalent conditions on the continuity and the marginals of a process to guarantee that such process is a Brownian motion. Using the intuition on martingales that we have developed in the previous sections we are now ready to give a different set of conditions that allow to draw the same conclusion:

THEOREM 5.1 (Lévy-Doob). *If $X(t)$ is a continuous martingale such that*

i) $X(0) = 0$,

ii) $X(t)$ *is a square integrable-martingale with respect to the filtration it generates,*

iii) $X(t)^2 - t$ *is a square integrable-martingale with respect to the filtration it generates*

then $X(t)$ is a standard Brownian motion.

It is important the $X(t)$ be continuous. For example if N_t is a jump process $N_t - t$ and $(N_t - t)^2 - t$ are both martingales but $N_t - t$ is quite different from Brownian motion.

PROOF. Our proof essentially follows that of Doob found in [**2**], which approaches the problem as a central limit theorem, proved through a clever trick using a telescopic sum. Fix a positive integer N and an $\varepsilon > 0$. Define

$$\tau(\varepsilon, N) = \inf\{s > 0 : \sup_{\substack{s_1<s_2<s\\|s_1-s_2|<1/N}} |X(s_1) - X(s_2)| = \varepsilon\}\ .$$

If there is no such time s, set $\tau = \infty$. Fix a time t. We what to show that the random variable $X(t)$ has the same Gaussian distribution as B_t. To do this it is enough to show that $\mathbb{E}e^{i\alpha X(t)} = e^{-\alpha^2 t/2}$, that is show that they both have the same characteristic functions (Fourier transform). It is a standard result in basic probability that if a sequence of random variables have characteristic functions which converge for each α to $e^{-\alpha^2 t/2}$ then the sequence of random variables has a limit and it is Gaussian. See [**1**] for a nice discussion of characteristic functions and convergence of probability measures. Hence, we will show that

$$\mathbb{E}\left\{e^{i\alpha X(t\wedge\tau)}\right\} \to e^{-\frac{\alpha^2 t}{2}} + O(\varepsilon) \text{ as } N\to\infty \text{ for any } \varepsilon > 0.$$

Since ε will be arbitrary and the left hand side is independent of ε, this will imply the result.

Partition the interval $[0,t]$ with point $t_k = \frac{kt}{N}$. Set

$$I = \left|\mathbb{E}\left\{\prod_{j=1}^{N} e^{i\alpha(X_j - X_{j-1})} - \prod_{j=1}^{N} e^{-\frac{\alpha^2}{2}(t_j - t_{j-1})}\right\}\right|$$

where $X_j := X(t_j\wedge\tau)$. In general, observe that the following identity holds

$$\begin{aligned} A_1A_2A_3\cdots A_N - B_1B_2\cdots B_N =& A_1A_2\cdots A_{N-1}(A_N - B_N)\\ &+A_1A_2\cdots A_{N-2}(A_{N-1}-B_{N-1})B_N\\ &+A_1A_2\cdots A_{N-3}(A_{N-2}-B_{N-2})B_{N-1}B_N\\ &\vdots\\ &+(A_1-B_1)B_2B_3\cdots B_N\ . \end{aligned}$$

Hence

$$I = \left|\mathbb{E}\left\{\sum_{k=1}^{N}\prod_{j=1}^{N-k-1} e^{i\alpha(X_j-X_{j-1})}\left(e^{i\alpha(X_{N-k}-X_{N-k+1})} - e^{-\frac{\alpha^2}{2}(t_{N-k}-t_{N-k+1})}\right)\prod_{j=N-k+1}^{N} e^{-\frac{\alpha^2}{2}(t_j-t_{j-1})}\right\}\right|$$

All of the terms in the first product have modulus one and all of the terms in the second product are less than one. Hence

$$I \leqslant \mathbb{E}\left\{\sum_{k=1}^{N}\left|\mathbb{E}\left\{e^{i\alpha(X_{N-k}-X_{N-k+1})} - e^{-\frac{\alpha^2}{2}(t_{N-k}-t_{N-k+1})}\Big|\mathcal{F}_{t_{N-k}\wedge\tau}\right\}\right|\right\}$$

Now observe that by Taylor's theorem

$$e^{i\alpha\Delta_k X} - e^{-\frac{\alpha^2}{2}\Delta_k t} = i\alpha\Delta_k X - \frac{\alpha^2}{2}(\Delta_k X)^2 + \frac{\alpha^2}{2}\Delta_k t + O(\Delta_k X)^3 + O(\Delta_k t)^2$$

where $\Delta_k X = X_k - X_{k-1}$ and $\Delta_k t = t_k - t_{k-1}$. The constants implied by $O(\Delta_k X)^3$ and $O(\Delta_k t)^2$ can be taken to be uniformly bounded for $\varepsilon\in(0,\varepsilon_0]$ and $N\in[N_0,\infty)$ for some $\varepsilon_0>0$ and $N_0<\infty$. Observe that by using the martingale assumptions on X and the optional stopping lemma, we have that

$$\begin{aligned} \mathbb{E}\big\{\Delta_{N-k}X|\mathcal{F}_{t_{N-k}\wedge\tau}\big\} &= 0\\ \mathbb{E}\big\{(\Delta_{N-k}X)^2|\mathcal{F}_{t_{N-k}\wedge\tau}\big\} &= \Delta_{N-k}(t\wedge\tau) \leqslant t_{N-k} - t_{N-k-1}\ . \end{aligned}$$

Here $\Delta_k(t\wedge\tau) = t_k\wedge\tau - t_k\wedge\tau$. By our definition of τ, $|\Delta_{N-k}X|\leqslant\varepsilon$. So we have

$$\mathbb{E}\big\{|\Delta_{N-k}X|^3|\mathcal{F}_{t_{N-k}\wedge\tau}\big\} \leqslant \big(\sup_{k,\omega}|\Delta_{N-k}X|\big)\mathbb{E}\big\{(\Delta_{N-k}X)^2|\mathcal{F}_{t_{N-k}\wedge\tau}\big\} \leqslant \varepsilon\mathbb{E}\big\{(\Delta_{N-k}X)^2|\mathcal{F}_{t_{N-k}\wedge\tau}\big\}\ .$$

And thus,

$$I \leqslant \mathbb{E}\left\{\sum_{k=1}^{N}\left|\mathbb{E}\{i\alpha\Delta_k X|\mathcal{F}_{t_{N-k}}\} - \mathbb{E}\{\frac{\alpha^2}{2}(\Delta_k X)^2|\mathcal{F}_{t_{N-k}}\} + \frac{\alpha^2}{2}\Delta_k t + C(\Delta_k t)^2 + C\mathbb{E}\{(\Delta_k X)^3|\mathcal{F}_{t_{N-k}\wedge\tau}\}\right|\right\}$$

$$\leqslant \left[-N\left(\frac{t}{N}\right)\frac{\alpha^2}{2} + N\left(\frac{t}{N}\right)\frac{\alpha^2}{2} + CN\left(\frac{t}{N}\right)^2 + C\varepsilon N\left(\frac{t}{N}\right)\right] = C\frac{t}{N} + \varepsilon Ct$$

Observe that $\tau \to \infty$ as $N \to \infty$ for any fixed ε. Hence we have that

$$\left|\mathbb{E}\left\{e^{i\alpha X(t)}\right\} - e^{-\frac{\alpha^2 t}{2}}\right| \leqslant \lim_{N\to\infty} I \leqslant \varepsilon Ct$$

Notice that the left hand side is independent of ε. Since C and t are fixed and ε was any arbitrary number in $(0, \varepsilon_0]$, we conclude that

$$\mathbb{E}\left\{e^{i\alpha X(t)}\right\} = e^{-\frac{\alpha^2 t}{2}}$$

□

We now give a slightly different formulation of the Levy-Doob theorem. Let M_t be a continuous martingale. Then by Theorem 4.2 if $[M]_t = t$ condition ii) of Theorem 5.1 is satisfied and we obtain the following result.

THEOREM 5.2 (Levy-Doob theorem). *If M_t is a continuous martingale with $[M]_t = t$ and $M_0 = 0$ then M_t is standard Brownian motion.*

6. Random time changes

Let M_t be a contiguous martingale with respect to the filtration $\mathcal{F}_t$. Since the quadratic variation map $t \mapsto [M]_t$ is non-decreasing, we can define its left-inverse by

$$\tau_t = \inf\{s \geqslant 0 : [M]_s \geqslant t\} \tag{8.10}$$

and the limiting value

$$[M]_\infty = \lim_{t\to\infty}[M]_t$$

THEOREM 6.1 (Dambis-Dubins-Schwartz). *Let M_t, τ_t be as above. If $[M]_\infty > T$ then $B_t = M_{\tau_t}$ is a Brownian motion on the interval $[0, T]$ with respect to the filtration $\mathcal{G}_t = \mathcal{F}_{\tau_t}$. Conversely, there exists a standard Brownian motion B_t such that $M_t = B_{[M]_t}$ for $t \geqslant 0$. This result also holds when M_t is a continuous* local *martingale.*

REMARK 6.2. *Theorem 6.1 shows that any continuous martingale is just the time change of Brownian motion with $[M]_t$ giving the rate at which fluctuations are injected into the system. This intuition is particularly useful in finance, where $[M]_t$ can be thought of a measure of the volatility of the process.*

PROOF OF THEOREM 6.1. By the definition of τ_t as the left-inverse of the map $t \mapsto [M]_t$ we have that $[\hat{B}]_t = [M]_{\tau_t} = t$. Hence $M_{\tau_t}^2 - t$ is a martingale. By localizing the stopping time τ_t to $\tau_t \wedge s$ for a finite s if necessary (*i.e.*, to allow for the application of the optional stopping theorem) we have that

$$\mathbb{E}(\hat{B}_t|\mathcal{G}_s) = \mathbb{E}(M_{\tau_t}|\mathcal{F}_{\tau_s}) = M_{\tau_s} = \hat{B}_s$$

and consequently we see that $\hat{B}_t$ is also a martingale. Hence the by the Levy-Doob Theorem (Theorem 5.2), $\hat{B}_t$ is a standard Brownian motion. The converse result follows from the first: for $\hat{B}_t$ defined above we see by the definition of τ_t that $\hat{B}_{[M]_t} = M_{\tau_{[M]_t}} = M_t$ since $\tau_{[M]_t} = t$. □

For martingales that can be written as

$$\mathrm{d}M_t = H_t \, \mathrm{d}B_t \,,$$

we know that $[M]_t = \int_0^t H_s^2 \, \mathrm{d}s$. Therefore, by the above theorem if $\int_0^\infty H_s^2 \, \mathrm{d}s = \infty$ we can write M_t as

$$M_t(\omega) = \hat{B}\left(\omega, \int_0^t H_s^2(\omega) \, \mathrm{d}s\right) \,, \tag{8.11}$$

for a Brownian motion $\hat{B}(\omega, s)$ that can be constructed from M_t. We note that we can explicitly invert the time-change: letting $f(t,\omega) = \int_0^t H_s(\omega)^2 \, \mathrm{d}s$ we have

$$H_t(\omega) = \sqrt{\partial_t f(t,\omega)}$$

This implies that M_t, *i.e.*, the time-changed Brownian motion $\hat{B}(\omega, f(t,\omega))$ satisfies the SDE

$$\mathrm{d}M_t = \mathrm{d}\hat{B}(f(t)) = \sqrt{\partial_t f(t)} \, \mathrm{d}B_t \,, \tag{8.12}$$

where, in general, $B \neq \hat{B}$! We also note that, if $H_s(\omega) = H_s$ *i.e.*, H_s is a deterministic process, the time-change is deterministic and the interpretation of the above calculation simplifies (cfr the next example). Furthermore, in this case, changing time for another Brownian motion $\tilde{B}$ still satisfies (8.11) *in distribution.*

Example 6.3. *We consider the time-change $H_s = \sigma e^{\alpha s}$ i.e.,*

$$f(t) = \int_0^t \sigma^2 e^{2\alpha s} = \sigma^2 \frac{e^{2\alpha t} - 1}{2\alpha} \,.$$

Then we have that the process $\hat{B}(f(t))$ is the (weak) solution to the SDE $\mathrm{d}X_t = \sigma e^{\alpha t} \, \mathrm{d}B_t$. *Now, consider the process*

$$U_t := e^{-\alpha t} X_t = e^{-\alpha t} \hat{B}\left(\sigma^2 \frac{e^{2\alpha t} - 1}{2\alpha}\right) \,.$$

By Itô's product rule we see that this process satisfies

$$\mathrm{d}U_t = -\alpha U_t \, \mathrm{d}t + \sigma \, \mathrm{d}B_t \,,$$

which is the well know SDE *for the Ornstein-Uhlenbeck process (cfr. Langevin equation).*

Time Change for an SDE. We now extend the above reasoning and use it to construct a new way of solving SDEs.
Consider the simple one dimensional SDE

$$dX_t = \sigma(X_t) \, \mathrm{d}B_t$$

with $\sigma(x) > 0$. We can rewrite the above equation as

$$\mathrm{d}B_t = \frac{1}{\sigma(X_t)} \mathrm{d}X_t \,.$$

Now, by Theorem 6.1 we write X_t as a time-changed Brownian motion $X_t(\omega) = \hat{B}(\omega, [X]_t)$, and in the new timescale τ defined by $[X]_t$ we have that the SDE reads

$$\mathrm{d}M_\tau = \frac{1}{\sigma(\hat{B}_\tau)} \mathrm{d}\hat{B}_\tau \,.$$

In the following, by abuse of notation we will denote the new timescale as the old one, *i.e.*, $\tau = t$. We assume that $\int_0^\infty \sigma^{-2}(\hat{B}_s) ds = \infty$ almost surely (A simple condition which ensures this is $|\sigma(x)| \leqslant c < \infty$ for all x.) Now we would like to invert the change of time we just performed *i.e.*, go back to the timescale where $M.$ is a Brownian motion. Similarly to the previous paragraph, we do this by defining the inverse transformation:

$$[M]_t = \int_0^t \sigma^{-2}(\hat{B}_s)ds =: G(t) \qquad \text{and} \qquad \tau_t = G^{-1}(t) = \inf\{s : [M]_s > t\} . \tag{8.13}$$

In other words, we are now in the same setting as in the previous section, where $f(t) = G^{-1}(t)$. At the same time, by the inverse function theorem we obtain

$$f'(t) = \partial_t G^{-1}(t) = \left(G'(G^{-1}(t))\right)^{-1} = \left(\frac{1}{\sigma\left(\hat{B}(G^{-1}(t))\right)^2}\right)^{-1} = \sigma(\hat{B}(\tau_t))^2$$

Inserting this into (8.12) we finally obtain

$$\mathrm{d}X_t = \mathrm{d}\hat{B}_{\tau_t} = \sigma(\hat{B}_{\tau_t})\,\mathrm{d}B_t = \sigma(X_t)\,\mathrm{d}B_t \tag{8.14}$$

Remark 6.4. *We note that the above calculation can be performed for a general choice of time-change*

$$H(t) = \int_0^t h(X_s)\,\mathrm{d}s\,, \quad \textit{and} \quad \tau_t = H^{-1}(t)\,,$$

resulting in the SDE *for* $Y_t = X_{\tau_t}$ *given by*

$$\mathrm{d}Y_t = \frac{\sigma(Y_t)}{\sqrt{h(Y_t)}}\,\mathrm{d}B_t\,,$$

which for the choice of $H = \sigma^2$ *gives the standard Brownian motion as a solution. Inverting this time transformation as done above will give the solution to the original* SDE*. Note again that the time-changed Brownian motion is a* weak *solution to the original* SDE*, since we first choose a Brownian motion according to which we solve in the* SDE *in the new timescale, and then we transform it back to the original timescale, mapping the solution to another Brownian motion.*

Now consider the full-fledged SDE

$$dX_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t \tag{8.15}$$

where $\mu : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathbb{R}^{m,d}$ and B_t is a m-dimensional Brownian motion. As we have done is Remark 6.4, we define the time change

$$H(t) := \int_0^t h(X_s)ds \qquad \text{and} \qquad \tau_t = H^{-1}(t) = \inf\{s : H(s) > t\} . \tag{8.16}$$

Then we can show the following result:

Theorem 6.5. *Let* X_t *be the solution to (8.15), then the process* $Y_t = X_{\tau_t}$ *is a weak solution to the* SDE

$$\mathrm{d}Y_t = \frac{\mu(Y_t)}{h(Y_t)}\,\mathrm{d}t + \frac{\sigma(Y_t)}{\sqrt{h(Y_t)}}\,\mathrm{d}B_t\,.$$

Proof. With the same definitions as above define $dM_t = \sqrt{h(X_t)}\,\mathrm{d}B_t$ and $B_t = M_{\tau_t}$. Since $[M]_t = \int_0^t h(X_s)\,\mathrm{d}t$ we see that B_t is a standard Brownian motion. Observe that $d\tau_t = h(X_{\tau_t})^{-1}\,\mathrm{d}t$ and

$$\mathrm{d}B_t = \frac{1}{\sqrt{h(X_{\tau_t})}}\,\mathrm{d}B_{\tau_t}$$

Defining $Y_t = X_{\tau_t}$, we have that

$$
\begin{aligned}
dY_t = \frac{1}{h(X_{\tau_t})} dX_{\tau_t} &= \frac{1}{h(X_{\tau_t})} \big(\mu(X_{\tau_t})\,\mathrm{d}t + \sigma(X_{\tau_t})\,\mathrm{d}B_{\tau_t}\big) \\
&= \frac{\mu(Y_t)}{h(Y_t)}\,\mathrm{d}t + \frac{\sigma(Y_t)}{\sqrt{h(Y_t)}} \frac{1}{\sqrt{h(X_{\tau_t})}} \mathrm{d}B_{\tau_t} \\
&= \frac{\mu(Y_t)}{h(Y_t)}\,\mathrm{d}t + \frac{\sigma(Y_t)}{\sqrt{h(Y_t)}}\,\mathrm{d}B_t
\end{aligned}
$$

□

Note that, similarly to all the cases above, it is only a weak solution since the Brownian motion B_t was constructed at the same time as the solution Y_t. A strong solution required that the Brownian motion be specified in advance.

EXAMPLE 6.6. *We consider the equation for the squared Bessel process (cfr problem sets)*

$$dX_t = \delta dt + 2\sqrt{X_t} dB_t$$

and define the time change

$$\tau = \frac{\sigma^2}{2\nu(2-\delta)} \left(1 - \exp\left(-\frac{2\nu t}{2-\delta}\right)\right) .$$

Then by the above theorem we obtain

$$d\tilde{X}_t = \delta\tau'(t)dt + 2\sqrt{\tilde{X}_t}\sqrt{\tau'(t)}dB(t) .$$

Now defining

$$Y_t = exp(\nu t)\tilde{X}_t^{1-\delta/2} ,$$

we have that

$$dY_t = \nu Y_t dt + \exp(\nu t)(1-\frac{\delta}{2})\tilde{X}_t^{-\delta/2} d\tilde{X}_t + \exp(\nu t)\left(2(-\frac{\delta}{2})(1-\frac{\delta}{2})\tau'(t)\tilde{X}_t^{-\delta/2}\right) dt .$$

and combining with the definition of τ and $\mathrm{d}\tilde{X}_t$ we obtain that $Y_t = X_\tau$ solves

$$dY_t = \nu Y_t dt + \sigma Y_t^{\frac{1-\delta}{2-\delta}} dW_t$$

REMARK 6.7. *The same argument shows that if*

$$dX_t = h_t \mu(X_t)\,\mathrm{d}t + \sqrt{h_t}\sigma(X_t)\,\mathrm{d}B_t$$

for some positive, adapted stochastic process h_t, then if $\tau_t = \int_0^t h_s^{-1} ds$ and $Y_t = X_{\tau_t}$ we have

$$dY_t = \mu(Y_t)\,\mathrm{d}t + \sigma(Y_t) d\hat{B}_t$$

for the standard Brownian motion $\hat{B}_t = M_{\tau_t}$ where $M_t = \int_0^t h_s dB_s$.

7. Martingale inequalities

We now present some very useful inequalities that allow to control the fluctuations of martingales. The first result is due to Doob and controls the probability distribution of the maximum of a martingale on a certain time interval. For this reason these inequalities are sometimes called Doob's maximal inequalities. The first one bounds from above the probability that the supremum of a martingale in an interval exceeds a certain a certain value λ, while the second bounds the first moment of such distribution, *i.e.*, the expected value of the supremum on the given interval.

Theorem 7.1 (Doob's Martingale Inequality). *Let M_t be a martingale (or a positive submartingale) with respect to the filtration $\mathcal{F}_t$. Then for $T > 0$ and for all $\lambda > 0$*

$$\mathbb{P}\left[\sup_{0\leqslant t\leqslant T} |M_t| \geqslant \lambda\right] \leqslant \frac{\mathbb{E}[|M_T|^p]}{\lambda^p} \qquad \textit{for all } p \geqslant 1\,,$$

and

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |M_t|\right] \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_T|^p] \qquad \textit{for all } p > 1\,.$$

Before turning to the proof, we remark the similarity of the first inequality with Markov's inequality, *i.e.*, given a random variable X, for every $p \geqslant 1$ we have

$$\mathbb{P}\left[|X| > \lambda\right] \leqslant \frac{\mathbb{E}\left[|X|^p\right]}{\lambda^p}\,.$$

The difference of the two inequalities is the supremum, under the condition of the process M_t being a martingale, in Doob's inequality.

Proof. First of all we note that by convexity of $|x|$ and x^p on $\mathbb{R}_+$ the process $|M_t|^p$ is a submartingale. Consequently defining the stopping time

$$\tau_\lambda := \inf\{t \;:\; |M_t| > \lambda\}\,,$$

we have by Doob's optional stopping theorem

$$\mathbb{E}\left[|M_{\tau_\lambda\wedge t}|^p\right] \leqslant \mathbb{E}\left[|M_t|^p\right]\,. \tag{8.17}$$

At the same time, we have that

$$\begin{aligned}\mathbb{E}\left[|M_{\tau_\lambda\wedge t}|^p\right] &= \mathbb{E}\left[|M_{\tau_\lambda\wedge t}|^p \mathbb{1}_{\tau_\lambda\leqslant t}\right] + \mathbb{E}\left[|M_{\tau_\lambda\wedge t}|^p \mathbb{1}_{\tau_\lambda > t}\right]\\ &= \lambda\mathbb{P}\left[\tau_\lambda \leqslant t\right] + \mathbb{E}\left[|M_t|^p \mathbb{1}_{\tau_\lambda > t}\right]\,.\end{aligned} \tag{8.18}$$

Combining (8.17) and (8.18) we finally obtain

$$\mathbb{P}\left[\sup_{s\in[0,t]} |M_s| \geqslant \lambda\right] = \mathbb{P}\left[\tau_\lambda \leqslant t\right] \leqslant \frac{\mathbb{E}\left[|M_t|^p \mathbb{1}_{\tau_\lambda\leqslant t}\right]}{\lambda^p} \leqslant \frac{\mathbb{E}\left[|M_t|^p\right]}{\lambda^p}$$

where in the last passage we have used the nonnegativity of $|M_t|$. □

The above result is key to derive numerous results in stochastic calculus. We have seen one example in the proof of Theorem 7.2. We can also use it to bound the supremum of Itô integrals:

Example 7.2. *Under the assumption that $\sigma_s \leqslant M < \infty$ we have shown in Section 3 that $\mathbb{E}\left[|\int_0^t \sigma_s \,\mathrm{d}B_s|^p\right] < \infty$. Consequently, by Doob's inequality (recall that for a martingale M_t, $|M_t|^p$ is a positive submartingale for $p \geqslant 1$) we have*

$$\mathbb{E}\left[\sup_{t\in(0,T)} |\int_0^t \sigma_s \,\mathrm{d}B_s|^p\right] \leqslant C_2\mathbb{E}\left[|\int_0^t \sigma_s \,\mathrm{d}B_s|^{2p}\right] < \infty\,.$$

We now introduce the very useful Burkholder-Davis-Gundy inequalities.

Theorem 7.3 (Burkholders-Davis-Gundy Inequality). *Let X_t be a local martingale, then for any $p \geqslant 1$*

$$c_p\mathbb{E}\big[[X_t]^p\big] \leqslant \mathbb{E}\Big[\sup_{0\leqslant s\leqslant t} |X_s|^{2p}\Big] \leqslant C_p\mathbb{E}\big[[X_t]^p\big]$$

where c_p, C_p are constants independent of the process, depending only on p.

PROOF. We only prove the upper bound of this result, under the simplifying assumption that $X_t = \int_0^t f_s(\omega) dB_s$ for a bounded process $f_s \leqslant M$ for $M < \infty$. For the complete versions of the proof see [**4, 8, 15**].
Doob's L^p maximal inequality combined with Itô's formula implies that

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |X_s|^{2p}\right] \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_t|^{2p}\right] \tag{8.19}$$
$$= \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[\frac{2p(2p-1)}{2}\int_0^t |X_s|^{2(p-1)} f(s)^2 ds + 2p\int_0^t |X_s|^{2p-1} f(s) dB_s\right].$$

Next we introduce the stopping time

$$\tau_N = \inf\{t \geqslant 0 : \int_0^t |X_s|^{4p-2}|f_s|^2 ds \geqslant N\}$$

Let $I_N(t) = 2p\int_0^t |X_{s\wedge\tau_N}|^{2p-1} f_{s\wedge\tau_N} dB_s$. Since the integrand is bounded by the construction of τ_N we have that $\mathbb{E} I_N(t) = 0$. Notice that

$$\mathbb{E}\left[2p\int_0^{t\wedge\tau_N} |X_s|^{2p-1} f(s) dB_s\right] = \mathbb{E}[I_N(t\wedge\tau_N)] = 0$$

where the last equality follows from the Optional Stopping Theorem. Next observe that

$$\mathbb{E}\left[\int_0^t |X_s|^{2(p-1)} f(s)^2 ds\right] \leqslant \mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |X_s|^{2(p-1)} \int_0^t f(s)^2 ds\right] = \mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |X_s|^{2(p-1)} [X_t]\right]$$

and by Holder's inequality with powers $p = p$ and $q = p/(p-1)$ we have that

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |X_s|^{2(p-1)} [X_t]\right] \leqslant \mathbb{E}\left[\sup_{0\leqslant s\leqslant t} |X_s|^{2p}\right]^{1-\frac{1}{p}} \mathbb{E}\Big[[X]_t^p\Big]^{\frac{1}{p}}$$

Putting everything together produces

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant t\wedge\tau_N} |X_s|^{2p}\right] \leqslant C\mathbb{E}\left[\sup_{0\leqslant s\leqslant t\wedge\tau_N} |X_s|^{2p}\right]^{1-\frac{1}{p}} \mathbb{E}\Big[[X]_{t\wedge\tau_N}^p\Big]^{\frac{1}{p}}$$

By the definition of the stopping time everything is finite, hence we can divide thought by the first term on the right to obtain

$$\mathbb{E}\left[\sup_{0\leqslant s\leqslant t\wedge\tau_N} |X_s|^{2p}\right]^{\frac{1}{p}} \leqslant C\mathbb{E}\Big[[X]_{t\wedge\tau_N}^p\Big]^{\frac{1}{p}}.$$

We realize that both right- and left hand side are uniformly bounded, under our assumption, by (8.5) and by $\int_0^t f_s^2 \mathrm{d}s < tM^2$ respectively. The proof is concluded by raising both sides to the power p and, by means of dominated convergence theorem, removing the stopping time by taking the limit as $N \to \infty$. $\square$

8. Martingale representation theorem

We conclude this chapter by introducing a last fundamental result about martingales, strengthening the connection between martingales and Ito integrals. Recall that Itô integrals of square-integrable processes are martingales. The Martingale Representation theorem, a quite remarkable result, essentially establishes that the converse result is also true: every martingale can be expressed as the Itô integral of a square-integrable process. Furthermore, such process is *unique* among the family of predictable processes. As suggested by the name, predictable processes are those whose value at time t can be *predicted* given the information before time t. Examples of such processes are given

by processes that are continuous from the left, *i.e.*, for which $\lim_{s\uparrow t} X_s = X_t$.
A precise definition of this class of processes is given below:

DEFINITION 8.1. *Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geqslant 0})$, then a continuous-time stochastic process $\{X_t\}_{t\geqslant 0}$ is predictable if X, considered as a mapping from $\Omega \times \mathbb{R}_+$, is measurable with respect to the σ-algebra generated by all left-continuous adapted processes.This σ-algebra is also called the predictable σ-algebra.*

One can think about predictable processes as processes that an external observer can control, as exemplified below:

REMARK 8.2. *This example lives in discrete time, where predictability implies that $X_{n+1} \in \mathcal{F}_n$. Suppose we have a certain amount of money V_n at a certain time t_n. We decide to invest a certain percentage X_n of this money in a title with value S_n at time t_n and put the remaining part $1 - X_n$ in our bank account. S_n can be modeled as a random variable, but so can X_n: our fund's allocation varies based on how the title's value fluctuates. Seeing the σ algebra $\mathcal{F}_n$ as information from the values of S_n up to time t_n. What makes S_n and X_n different is that we have control of the amount of money X_{n+1} that we want to invest at the time t_n in the title S_n because this decision must be made before t_{n+1}. In other words, the value of X_{n+1} must depend exclusively the information up to time t_n, i.e., $X_{n+1} \in \mathcal{F}_n$.*

THEOREM 8.3 (Martingale representation theorem). *Let M_t be a square-integrable [or local] F_t^B-martingale on $(0, T)$ (where possibly $T = \infty$) then there exists a square-integrable process C_s [or a process C_s s.t. $\mathbb{P}\left[\int_0^T C_s^2 \,\mathrm{d}s < \infty\right] = 0$] such that*

$$M_t = M_0 \int_0^t C_s \,\mathrm{d}B_s \,.$$

We do not prove the above result here, but refer to [**16**] for a proof. We note that the result is restricted to $\mathcal{F}_t^B$. This result is especially useful in finance, as we will see in the final chapters of this course.

CHAPTER 9

Girsanov's Theorem

1. An illustrative example

We begin with a simple example. We will frame it in a rather formal way as this will make the analogies with later examples clearer.

One-dimensional Gaussian case. Let us consider the probability space $(\omega, \mathbb{P}, \mathcal{F})$ where $\Omega = \mathbb{R}$ and $\mathbb{P}$ is the standard Gaussian with mean zero on variance one. (For completeness let $\mathcal{F}$ be the Borel σ-algebra on $\mathbb{R}$.). We define two random variables Z and $\tilde{Z}$ on this probability space. As always, a real valued random variable is a function from Ω into $\mathbb{R}$. Let us define

$$Z(\omega) = \omega \qquad \text{and} \qquad \tilde{Z}(\omega) = \omega + \mu$$

for some fixed constant μ. Since ω is drawn under $\mathbb{P}$ with respect the $\mathcal{N}(0,1)$ measure on $\mathbb{R}$ we have that Z is also distributed $\mathcal{N}(0,1)$ and $\tilde{Z}$ is distributed $\mathcal{N}(\mu, 1)$.

Now let us introduce the density function associated to $\mathbb{P}$ as

$$\phi(\omega) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{\omega^2}{2})$$

Now we introduce the function

$$\Lambda_\mu(\omega) = \frac{\phi(\omega - \mu)}{\phi(\omega)} = \exp\left(\omega\mu - \frac{\mu^2}{2}\right)$$

Since Λ_μ is a function from Ω to $\mathbb{R}$ is can be viewed as a random variable and we have

$$\mathbb{E}_{\mathbb{P}}\Lambda_\mu = \int_\omega \Lambda_\mu(\omega)\mathbb{P}(d\omega) = \int_{-\infty}^{\infty} \Lambda_\mu(\omega)\phi(\omega)d\omega = \int_{-\infty}^{\infty} \phi(\omega - \mu)d\omega = 1$$

since $\phi(\omega - \mu)$ is the density of a $\mathcal{N}(\mu, 1)$ random variable. Hence Λ_μ is a $L^1(\Omega, \mathbb{P})$ random variable. Hence we can define a new measure $\mathbb{Q}$ on Ω by

$$\mathbb{Q}(d\omega) = \Lambda_\mu(\omega)\mathbb{P}(d\omega).$$

This means that for any random variable X on Ω we have that the expected value with respect to the $\mathbb{Q}$, denoted by $\mathbb{E}_{\mathbb{Q}}$ is define by

$$\mathbb{E}_{\mathbb{Q}}[X] = \mathbb{E}_{\mathbb{P}}[X\Lambda_\mu]$$

Furthermore observe that for any bounded $f : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}} f(Z) = \mathbb{E}_{\mathbb{P}}[f(Z)\Lambda_\mu] &= \int_{-\infty}^{\infty} f(Z(\omega))\Lambda_\mu(\omega)\phi(\omega)d\omega = \int_{-\infty}^{\infty} f(\omega)\phi(\omega - \mu)d\omega \\ &= \int_{-\infty}^{\infty} f(\omega + \mu)\phi(\omega)d\omega = \mathbb{E}_{\mathbb{P}} f(\tilde{Z})\end{aligned}$$

Which implies that the distribution of Z under the measure $\mathbb{Q}$ is the same as the distribution of $\tilde{Z}$ under distribution $\mathbb{P}$.

Example 1.1 (Importance sampling). *Let $f : \mathbf{R} \to \mathbb{R}$, and let X be distributed $\mathcal{N}(\mu, 1)$. We have that*

$$\mathbb{E}f(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-\frac{(x-\mu)^2}{2}} dx$$

for some $\mu \in \mathbb{R}$.
For n large and $\{X_i\}_1^n$ iid $\mathcal{N}(0, 1)$, we estimate the above expected value by sampling, *i.e.,*

$$\mathbb{E}[f(X)] \approx \frac{1}{n} \sum_{i=1}^{n} f(X_i)$$

*The problem of the above method is that for not-so-large values of μ (*e.g., *$\mu > 6$), taking for example $f = \mathbb{1}_{X<0}$ we would need a very large amount of samples before sampling the tail of $\mathcal{N}(\mu, 1)$, i.e., elements that are relevant for our estimation.*
However, let Y be distributed $\mathcal{N}(0, 1)$. Then by the procedure outlined above we have:

$$\begin{aligned} \mathbb{E}[f(X)] =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{e^{-\frac{(x-\mu)^2}{2}}}{e^{-\frac{x^2}{2}}} e^{-\frac{x^2}{2}} dx = \mathbb{E}\left[f(Y) e^{\mu Y - \frac{\mu^2}{2}} \right] \\ \approx& \frac{1}{n} \sum_{i=1}^{n} f(Y_i) e^{\mu Y_i - \frac{\mu^2}{2}} \end{aligned}$$

for $\{Y_i\}_{i=1}^n$ iid $\mathcal{N}(0, 1)$. Under this new distribution, the indicator function often positively to the sampling, and we need significantly less samples to obtain an accurate estimate of the expectation.

Multidimensional Gaussian case. Now let's consider a higher dimensional version of the above example. Let $\Omega = \mathbb{R}^n$ and let $\mathbb{P}$ be n-dimensional Gaussian probability measure with covariance $\sigma^2 I$ where $\sigma > 0$ and I is the $n \times n$ dimensional covariance matrix. In analogy to before, we define for $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$

$$\phi(\omega) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\Big(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} \omega_i^2 \Big)$$

and for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$

$$\Lambda_\mu(\omega) = \frac{\phi(\omega - \mu)}{\phi(\omega)} = \exp\Big(\frac{1}{\sigma^2} \sum_{i=1}^{n} \omega_i \mu_i - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \mu_i^2 \Big)$$

Then if we define the $\mathbb{R}^n$ valued random variables $Z(\omega) = (Z_1(\omega), \ldots, Z_n(\omega))$ and $\tilde{Z}(\omega) = (\tilde{Z}_1(\omega), \ldots, \tilde{Z}_n(\omega)) = Z(\omega) + \mu$. Then if we define $\mathbb{Q}(d\omega) = \Lambda_\mu(\omega)\mathbb{P}(d\omega)$ then following the same reasoning as before that the distribution of Z under $\mathbb{Q}$ is the same as the distribution of $\tilde{Z}$ under $\mathbb{P}$.

2. Tilted Brownian motion

Consider the tilted Brownian motion process

$$\mathrm{d}X_t = \mu \,\mathrm{d}t + \mathrm{d}B_t \,,$$

where B_t is standard Brownian Motion, $\mu \in \mathbb{R}$. Furthermore, let $0 = t_0 < t_1 < \cdots < t_n \leqslant T$, and $f, g : \mathbb{R}^n \to \mathbb{R}$ such that $f(x_1, x_2, \ldots, x_n) = g(x_1, x_2 - x_1, \ldots, x_n - x_{n-1})$.

The to compute the expectation of f we write:

$$\begin{aligned}
&\mathbf{E}\ \left[g(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})\right] \\
&= \int_{\Omega\times\cdots\times\Omega} \left(\frac{g(x_1, x_2 - x_1, \ldots, x_n - x_{n-1})}{(2\pi)^{\frac{n}{2}} t_1^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}} \cdots (t_n - t_{n-1})^{\frac{1}{2}}} \right) \prod_{i=1}^{n} e^{-\frac{\left[(x_i - x_{i-1}) - \mu(t_i - t_{i-1})^2\right]^2}{2(t_i - t_{i-1})}} \prod \mathrm{d}x_i
\end{aligned}$$

In light of what has been discussed in the previous section, we transform the above in iid Gaussian distributions:

$$\begin{aligned}
\prod_{i=1}^{n} e^{-\frac{\left[(x_i - x_{i-1}) - \mu(t_i - t_{i-1})^2\right]^2}{2(t_i - t_{i-1})}} &= \prod_{i=1}^{n} e^{-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}} \prod_{i=1}^{n} e^{\mu(x_i - x_{i-1}) - \frac{1}{2}\mu^2(t_i - t_{i-1})} \\
&= e^{\mu x_n - \frac{1}{2}\mu^2 t_n} \prod_{i=1}^{n} e^{-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}} .
\end{aligned}$$

Now we can consider the multiplication as the desired measure of Gaussian distribution and the prefactor as the random variable $\Lambda_\mu(\omega, t)$:

$$\begin{aligned}
\mathbf{E}\left[f(X_{t_1}, \ldots, X_{t_n})\right] &= \mathbf{E}\left[f(B_{t_1}, \ldots, B_{t_n}) e^{\mu B_{t_n} - \frac{1}{2}\mu^2 t_n}\right] \\
&= \mathbf{E}\left[f(B_{t_1}, \ldots, B_{t_n}) \Lambda_\mu(\omega, t)\right]
\end{aligned}$$

We note *en passant* that the "coefficient" $\Lambda_\mu(\omega, t)$ can be written as a martingale $M_t(\omega)$, more precisely the exponential martingale $M_t = e^{\mu B_t - \frac{1}{2}\mu^2 t}$ (we are going to define this concept more precisely in the next section).

3. Girsanov's Theorem for SDEs

We now introduce some notation to generalize the above observations to the framework of measure theory. Let $(\Omega, \mathcal{F})$ be a measurable space, then

DEFINITION 3.1. *Given two measures μ, ν, we say that ν is absolutely continuous wrt (denoted by μ $\nu \ll \mu$ if*

$$\mu(A) = 0 \Rightarrow \nu(A) = 0 \quad \textit{for all measurable sets } A.$$

Provided that a measure $\mathbb{Q}$ is absolutely continuous WRT another measure $\mathbb{P}$, the following theorem from measure theory ensures that it is possible to perform the changes of measure that we carried out in the previous section, *i.e.*, it is possible to define a random variable Λ (the reweighting factor) that compensates for such change of measure.

THEOREM 3.2 (Radon Nikodym). *Let $\mathbb{P}, \mathbb{Q}$ be two probability measures on $(\Omega, \mathcal{F})$, such that $Q \ll P$, then there exists a measurable function $\Lambda : \Omega \mapsto \mathbb{R}$ (a random variable) such that $\mathbb{E}_{\mathbb{P}}[\Lambda] = 1$ and*

$$\mathbb{Q}[A] = \mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_A \Lambda\right] = \int_A \Lambda(\omega)\, \mathrm{d}\mathbb{P}(\omega) \qquad \forall\, A \in \mathcal{F} .$$

We denote such function

$$\Lambda(\omega) = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\omega) ,$$

and we refer to it as the Radon Nikodym derivative*.*

The assumption of absolute continuity guarantees that the Radon Nikodym derivative is well defined. Indeed, in the case where both probability measures have densities $\rho_{\mathbb{P}}$, $\rho_{\mathbb{Q}}$, $\Lambda = \rho_{\mathbb{Q}}/\rho_{\mathbb{P}}$ and absolutely continuity guarantees that the above ratio is well defined (*i.e.*, it does not explode).

We now present, without proof, a lemma from measure theory that allows to obtain most of the results in this chapter.

LEMMA 3.3 (General Bayes rule). *Let μ and ν be probability measures on $(\Omega, \mathcal{F})$ with $d\nu(\omega) = f(\omega)d\mu(\omega)$ for some $f \in L^1(\mu)$. Let X be a random variable with:*

$$\mathbf{E}_\nu|X| = \int |X(\omega)| d\nu(\omega) = \int |X(\omega)| f(\omega) d\mu(\omega)$$

If $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, then:

$$\mathbf{E}_\nu \left[X|\mathcal{G}\right] \mathbf{E}_\mu \left[f|\mathcal{G}\right] = \mathbf{E}_\mu \left[fX|\mathcal{G}\right]$$

Before using the above theorem in the context of stochastic processes, we recall the concept of stochastic exponential of a process X_t, given by

$$\mathcal{E}(X)_t = \exp\left(X_t - X_0 - \frac{1}{2}[X]_t\right) .$$

Recall that the stochastic integral of a process X_t are defined as the solution to the abbrsde

$$\mathrm{d}U_t = U_t \, \mathrm{d}X_t \, . \tag{9.1}$$

When X_t wr know by the Martingale representation theorem Theorem 8.3 that we can express $\mathrm{d}X_t = C_s \, \mathrm{d}B_t$ for a predictable process C_s. Therefore, by (9.1) stochastic exponentials of local martingales are local martingales themselves, as summarized in the following theorem. This result also gives a sufficient condition (called the Novikov condition) for the stochastic exponential of a (local) martingale to be a true martingale.

THEOREM 3.4 (Exponential Martingale). *If M_t is a local martingale with $M_0 = 0$ (like, for instance, every $\int_0^t a_s \, \mathrm{d}B_s$ with $\mathbb{P}\left[\int_0^t a_s^2 \, \mathrm{d}s < \infty\right] = 1$) then the stochastic exponential $\mathcal{E}(M)_t$ is a continuous positive local martingale, and hence a supermartingale. Furthermore, if*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}[M]_T\right)\right] < \infty \, , \tag{Novikov}$$

then $\mathcal{E}(M)_t$ is a martingale on $[0, T]$ with $\mathbb{E}\left(\mathcal{E}(M)_t\right) = 1$.

REMARK 3.5. *Other conditions guaranteeing that the stochastic exponential of a local martingale is a true martingale exist. Some of them are summarized in* [**9**, Thm. 8.14 – 8.17]*. Furthermore, if M_t has the form $M_t = \int_0^t a_s \, \mathrm{d}B_s$, then the condition $a_s \leqslant c(s) < \infty$ for all $\in (0, T)$ is a sufficient condition for $\mathcal{E}(M)_t$ to be a martingale.*

We finally come to the first version of Girsanov's theorem. This result allows to do something very similar to what was done in the first section of this chapter: Switching to a new probability measure so that an "unnatural" random variable becomes a normal-distributed one. This result can be generalized to the framework of stochastic processes: Girsanov's theorem allows, under some conditions summarized below, to transform an Itô process

$$\mathrm{d}Y_t = a_t(\omega) + \mathrm{d}B_t \tag{9.2}$$

on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the "simplest" stochastic process we encountered in this course, *i.e.*, Brownian motion, by changing the measure on that space.

THEOREM 3.6 (Girsanov I). *Let Y_t be defined as in (9.2) with B_t a Brownian motion under $\mathbb{P}$. Assume that $\int_0^t a_s dB_s$ is well defined, define the stochastic exponential*

$$\Lambda_t = \exp\left[-\int_0^t a_s\, \mathrm{d}B_s - \frac{1}{2}\int_0^t a_s^2\, \mathrm{d}s\right] .$$

and assume that Λ_t is a martingale on $[0,T]$ with respect to $\mathbb{P}$ (i.e., a $\mathbb{P}$-martingale). Then under the (equivalent) probability measure

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}(\omega) = \Lambda_T(\omega) \tag{9.3}$$

the process Y_t is a Brownian motion $\hat{B}_t$ on $[0,T]$.

PROOF. We want to show that Y_t is a SBM wrt $\mathbb{Q}$. To do so, by Lévy's characterization of Brownian motion Theorem 5.2, it is sufficient to show that

i) Y_t is a local martingale wrt $\mathbb{Q}$,
ii) $[Y]_t = t$,

provided that $Y_0 = 0$ (which we assume without loss of generality).

Part ii) follows from the following computation:

$$\mathrm{d}[\tilde{B}]_t = \mathrm{d}[Y]_t = (a_t\,\mathrm{d}t + \mathrm{d}B_t)\cdot(a_t\,\mathrm{d}t + \mathrm{d}B_t) = \mathrm{d}[B]_t = \mathrm{d}t\,,$$

provided that quadratic variation of processes are unchanged by absolutely continuous changes of probability measures such as the one defined by Λ_T. To show this, because the quadratic variation process is defined as a limit in probability, it is enough to show that for a sequence of random variables $\{X_n\}$, if $\lim^p_{n\to\infty} X_n = X$ in probability in $\mathbb{P}$ then the same holds in $\mathbb{Q}$. To this aim, let $A_n := \{|X_n - X| > \varepsilon\}$ and assume $\mathbb{P}[A_n] \to 0$ then by integrability of Λ_T we can apply dominated convergence theorem and obtain that

$$\mathbb{Q}[A_n] = \mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{A_n}\Lambda_T\right] \to 0\,.$$

For part i) we apply Itô's product rule to $K_t = Y_t\Lambda_t$ and obtain $dK_t = \Lambda_t dY_t + Y_t d\Lambda_t + dY_t d\Lambda_t$. Combining this with the SDE for Λ_T,

$$d\Lambda_t = -\Lambda_t a_t dB_t$$

we obtain

$$\begin{aligned} dK_t =& \Lambda_t(a_t\,\mathrm{d}t + dB_t) - Y_t\Lambda_t a_t dB_t - \Lambda_t a_t dB_t^2 \\ =&\Lambda_t(a_t\,\mathrm{d}t + dB_t) - Y_t\Lambda_t a_t dB_t - \Lambda_t a_t\,\mathrm{d}t \\ =&\Lambda_t(1 - Y_t a_t)dB_t \end{aligned}$$

and so K_t is a martingale wrt $\mathbb{P}$.
Now, we have that

$$\mathbb{E}_{\mathbb{Q}}\left[Y_t|\mathcal{F}_s\right] = \frac{\mathbb{E}_{\mathbb{P}}\left[\Lambda_t Y_t|\mathcal{F}_s\right]}{\mathbb{E}_{\mathbb{P}}\left[\Lambda_t|\mathcal{F}_s\right]} = \frac{K_s}{\Lambda_s} = Y_s\,,$$

implying that Y_s is a martingale wrt $\mathbb{Q}$. □

REMARK 3.7. *We note that instead of proving part ii) of the above theorem one could also have applied Theorem 5.1, i.e., we could have shown that $K_t^2 - t$ is a martingale. The proof of this result follows the same lines of the one of part i) above*

Example 3.8 (Brownian motion Tracking a Continuous Function). *We would like to estimate the probability that during the interval $[0,T]$ Brownian motion B_t stays in a "tube" of radius ε around a given differentiable function $h(t) \in C^1(\mathbb{R})$ with $h(0) = 0$. More precisely, we would like to estimate the following probability:*

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |B_t - h(t)| < \varepsilon\right) > 0$$

Let the event G be given by:

$$\begin{aligned} G = \ & \{|B_s - h(s)| < \varepsilon, s \in [0,1]\} \\ = \ & \{|X_s| < \varepsilon, s \in [0,1]\} \end{aligned}$$

for the process $X_s = B_s - h(s)$ which has differential

$$dX_s = -h'(s)ds + dB_s$$

Then by the above theorem we define the change of measure

$$\Lambda_t = e^{\left(\int_0^t h'(s)dB_s - \frac{1}{2}\int_0^t |h'(s)|^2 ds\right)}$$

Because $h'(s)$ is continuous on a compact interval it is uniformly bounded and the Novikov condition holds. Hence we can define the measure $d\mathbb{Q} = \Lambda_t d\mathbb{P}$, by the above theorem under $\mathbb{Q}$, X_t is a standard BM. Therefore we can write

$$\mathbb{Q}(G) = \int \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \mathbb{1}_G d\mathbb{P} \leqslant \left(\int \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2 d\mathbb{P}\right)^{\frac{1}{2}} \mathbb{P}(G)^{\frac{1}{2}}$$

where in the step inequality we have used Cauchy-Schwartz inequality and so:

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant 1} |B_t - h(t)| < \varepsilon\right) \geqslant \frac{\mathbb{Q}(\sup_{(0,1)} |\hat{B}_s| < \varepsilon)^2}{\int \left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) d\mathbb{Q}}$$

Looking at the above inequality we see that we have reduced the estimation of the relevant probability to the estimation of the probability of Brownian motion exiting an interval and the expected value of the random variable Λ_1.

The above result can be extended to the d-dimensional setting with nontrivial diffusion coefficient $\sigma(X_t)$. Furthermore, we may be interested in transforming Y_t (in the distributional sense) to a different Itô process X_t different than Brownian motion. Conditions to do this are summarized in the following more general theorem:

Theorem 3.9 (Girsanov II). *Let $X_t, Y_t \in \mathbb{R}^d$ be processes satisfying*

$$\begin{aligned} \mathrm{d}X_t &= \mu(X_t,t)\,\mathrm{d}t && +\sigma(X_t,t)\,\mathrm{d}B_t\,, \\ \mathrm{d}Y_t &= (\mu(Y_t,t) + \gamma(\omega,t))\,\mathrm{d}t && +\sigma(Y_t,t)\,\mathrm{d}B_t\,, \end{aligned}$$

with $Y_0 = X_0 = x$ for a m-dimensional $\mathbb{P}$-Brownian motion B_t on $t \in [0,T]$. Suppose that there exists a process $u(\omega,t)$ such that

$$\sigma(Y_t)u(\omega,t) = \gamma(\omega,t)\,.$$

Furthermore let

$$\Lambda_t := \exp\left[-\int_0^t u(\omega,s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t u(\omega,s)^2\,\mathrm{d}s\right], \tag{9.4}$$

Then if Λ_t is a $\mathbb{P}$-martingale on $[0,T]$ and $\mathbb{Q}$ is defined as in (9.3) we have that

$$\mathrm{d}Y_t = \mu(Y_t,t)\,\mathrm{d}t + \sigma(Y_t,t)\,\mathrm{d}\hat{B}_t\,,$$

for a $\mathbb{Q}$-Brownian motion

$$\hat{B}_t = \int_0^t u(\omega, s)\, \mathrm{d}s + B_t\,.$$

PROOF. It follows from Theorem 3.6 that $\hat{B}_t$ is a Brownian motion wrt $\mathbb{Q}$. Furthermore we observe that

$$\begin{aligned}
\mathrm{d}Y_t &= (\mu(Y_t, t) + \gamma(\omega, t))\,\mathrm{d}t\, + \sigma(Y_t, t)(\,\mathrm{d}\hat{B}_t - u(\omega, t)\,\mathrm{d}t) \\
&= (\mu(Y_t, t) + \gamma(\omega, t))\,\mathrm{d}t\, + \sigma(Y_t, t)\,\mathrm{d}\hat{B}_t - \gamma(\omega, t)\,\mathrm{d}t \\
&= \mu(Y_t, t)\,\mathrm{d}t\, + \sigma(Y_t, t)\,\mathrm{d}\hat{B}_t
\end{aligned}$$

as desired. □

We note that the above result can be added to our arsenal of methods to find weak solutions to SDES! Indeed, let X_t, Y_t be defined by:

$$\begin{aligned}
☹ \quad dX_t &= \mu_1(X_t)\,\mathrm{d}t + \sigma(X_t)dB_t \\
☺ \quad dY_t &= \mu_2(Y_t)\,\mathrm{d}t + \sigma(Y_t)dB_t \\
X_0 &= Y_0 = x
\end{aligned}$$

and assume that we cannot solve ☹ but have an idea on how to solve ☺. Then we can define $u(y)$ by:

$$\sigma(y)u(y) = \mu_2(y) - \mu_1(y)$$

and set, as in Theorem 3.9

$$\Lambda_t = e^{-\int_0^t u(Y_s)dB_s - \frac{1}{2}\int_0^t |u(Y_s)|^2 ds}$$

which allows us to define the measure $d\mathbb{Q} = \Lambda_t d\mathbb{P}$. Then by Theorem 3.9 we have that

$$\hat{B}_t = B_t + \int_0^t u(Y_s)ds$$

is a standard Brownian motion under $\mathbb{Q}$, and

$$\begin{aligned}
dY_t &= \mu_1(Y_t)\,\mathrm{d}t + \sigma(Y_t)d\hat{B}_t \\
&= \mu_1(Y_t)\,\mathrm{d}t + \sigma(Y_t)\left[u(Y_t)\,\mathrm{d}t + dB_t\right] \\
&= \mu_1(Y_t)\,\mathrm{d}t + \mu_2(Y_t)\,\mathrm{d}t - \mu_1(Y_t)\,\mathrm{d}t + \sigma(Y_t)dB_t \\
&= \mu_2(Y_t)\,\mathrm{d}t + \sigma(Y_t)dB_t
\end{aligned}$$

Hence, Y_t in $\mathbb{Q}$ solves the same SDE as X_t, but with a different Brownian motion. This implies that the Law of Y_t on $C(0, T; \mathbb{R}^d)$ (and therefore all of its marginals) is equivalent to the Law of X_t on $C(0, T; \mathbb{R}^d)$. Hence we can write the unknown marginals for the process ☹ in $\mathbb{P}$ as

$$\mathbb{E}_{\mathbb{P}}\left[f(X_t)\right] = \mathbb{E}_{\mathbb{Q}}\left[f(Y_t)\right] = \mathbb{E}_{\mathbb{P}}\left[f(Y_t)\Lambda_T\right]\,,$$

i.e., as an expectation on a process that we know multiplied by a weighting factor that can be estimated/computed.

CHAPTER 10

One Dimensional SDEs

1. Natural Scale and Speed measure

We now want to consider SDEs which do not satisfy the Lipschitz assumptions of Chapter 3. Let b and σ be bounded, continuous real-valued functions with σ uniformly bounded from below by a positive constant. Consider the SDE

$$dX_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t \tag{10.1}$$

We want to find a function $\phi : \mathbb{R} \to \mathbb{R}$ so that if we define $Y_t = \phi(X_t)$ then Y_t is a martingale. Applying Itô's formula gives

$$dY_t = (L\phi)(X_t)\,\mathrm{d}t + \phi'(X_t)\sigma(X_t)\,\mathrm{d}B_t\,. \tag{10.2}$$

where L is the generator of the process X_t defined by

$$(L\phi)(x) = b(x)\phi'(x) + \frac{1}{2}\sigma^2(x)\phi''(x)$$

Assuming that our choice of ϕ is such that ϕ' is bounded, Y_t will be a martingale if $(L\phi)(x) = 0$. This implies that

$$(\log\phi')' = \frac{\phi''}{\phi'} = -\frac{2b}{\sigma^2} \quad \text{implies} \quad \phi(x) = \int_\alpha^x \exp\Big(-\int_\beta^y \frac{2b}{\sigma^2}(z)dz\Big)dy$$

for any choice of α and β. Notice that by construction ϕ is twice-differentiable, positive and monotone increasing function of $\mathbb{R}$ onto $\mathbb{R}$. Hence ϕ is invertible and we can understand ϕ as a warping of $\mathbb{R}$ so that X_t becomes a Martingale. For this reason, the function ϕ is called the *natural scale* for the process X_t.

In light of (10.2), $Y_t = \phi(X_t)$ satisfies

$$dY_t = (\phi'\sigma)(\phi^{-1}(Y_t))\,\mathrm{d}B_t \tag{10.3}$$

which shows that Y_t not only is a Martingale but it is again an SDE.

In the discussion of random time changes, we saw that the when the martingale M_t was solves the SDE

$$dM_t = g(M_t)\,\mathrm{d}B_t \tag{10.4}$$

then if we consider M_t on the time scale

$$\tau(t) = \int_0^t \frac{1}{g^2(M_s)}ds$$

then $B_t = M_{\tau(t)}$ is a Brownian motion. Since the rate of randomness injection into the system, as measured by the quadratic variation, for a Brownian motion is one, this time changes is given a distinguished status. The measure on $\mathbb{R}$ which gives this time change is $\frac{1}{g^2(x)}$ when integrated along the trajectory so called the *speed measure*. In the setting of (10.4), the speed measure, denoted $m(x)dx$, would be

$$m(x) = \frac{1}{g^2(x)} .$$

Returning to the setting with a drift term (10.1), we look for the time change of the resulting martingale after the system has been put on its natural scale. Looking at (10.3), we see that

$$\frac{1}{[(\phi'\sigma)(\phi^{-1}(y))]^2}dy \tag{10.5}$$

is the speed measure for the system expressed in the y variable where $y = \phi(x)$. Undoing this transform using and using $dy = \phi'(x)dx$ shows the speed measure in the original variables to be

$$m(x)dx = \frac{1}{(\phi'\sigma^2)(x)}dx$$

2. Existence of Weak Solutions

In the previous section we saw how to transform the one-dimensional SDE (10.1) in to a Brownian motion by warping space and changing time. Noticing that each of these processes was reversible/invertible, we now reverse our steps to turn a Brownian motion in to a solution of (10.1).

Let B_t be a standard Brownian motion. Looking back at (10.3) and (10.5), we define Y_t by

$$dY_t = (\phi'\sigma)(\phi^{-1}(Y_t))dB_t$$

The equation has a weak solution given by $Y_t = B_{T_t}$ where

$$T_t = \int_0^t \left[(\phi'\sigma)(\phi^{-1}(B_s))\right]^2 ds .$$

Next we define $X_t = \psi(Y_t)$ where for notation compactness we have defined $\psi = \phi^{-1}$ then Itô's formula tells up that

$$dX_t = \psi'(Y_t)dY + \frac{1}{2}\psi''(Y_t)d[Y]_t$$

`Need to finish argument`

3. Exit From an Interval

Let $M_t = \phi(X_t)$ where ϕ is the natural scale and X_t solves (10.1). And define the hitting time

$$\tau_y = \inf\{t \geqslant 0 : X_t = y\}$$

Assuming that $X_0 = x \in (a, b)$ we define the exit time of the interval by

$$\tau_{(a,b)} = \tau_a \wedge \tau_b .$$

By the construction of ϕ, M_t is a martingale. Hence since $\tau_{(a,b)} \wedge t$ is a bounded stopping time, the Optional Stopping lemma says that

$$\mathbb{E}_x M_{\tau_{(a,b)} \wedge t} = \mathbb{E}_x M_0 = \phi(x)$$

If we assume that $\sigma(y) > 0$ for all $y \in [a, b]$ then it is possible to show that

$$\mathbb{E}_x \tau_{(a,b)} < \infty .$$

This in turn implies that $\tau_{(a,b)} \wedge t \to \tau_{(a,b)}$ as $t \to \infty$. Hence we have that

$$\begin{aligned}\phi(x) =& \mathbb{E}_x M_{\tau_{(a,b)}} = \mathbb{P}_x(\tau_a \leqslant \tau_b)M_{\tau_a} + \mathbb{P}_x(\tau_b \leqslant \tau_a)M_{\tau_b}\\ =& \mathbb{P}_x(\tau_a \leqslant \tau_b)\phi(a) + (1 - \mathbb{P}_x(\tau_a \leqslant \tau_b))\phi(b)\end{aligned}$$

Rearranging produces

$$\mathbb{P}_x(\tau_a \leqslant \tau_b) = \frac{\phi(x) - \phi(a)}{\phi(b) - \phi(a)} \tag{10.6}$$

Another way to find this formula is to set $u(x) = \mathbb{P}_x(\tau_a \leqslant \tau_b)$. Then $u(x)$ solves the PDE

$$(Lu)(x) = 0 \qquad x \in (a,b), \quad u(a) = 1, \quad \text{and} \quad u(b) = 0$$

It is not heard to see that the above formula solves this PDE. (Try the case when X_t is a standard Brownian motion to get started).

Now we derive a formula for $v(x) = \mathbb{E}_x \tau_{(a,b)}$. Since it is a solution to

$$(Lv) = -1 \quad x \in (a,b) \qquad \text{and} \qquad v(a) = v(b) = 0$$

one finds

$$v(x) = \mathbb{E}_x \tau_{(a,b)} = 2\frac{\phi(x) - \phi(a)}{\phi(b) - \phi(a)} \int_x^b \big[\phi(b) - \phi(z)\big] m(z) dz + 2\frac{\phi(b) - \phi(x)}{\phi(b) - \phi(a)} \int_a^x \big[\phi(z) - \phi(a)\big] m(z) dz$$

4. Recurrence

DEFINITION 4.1. *A one-dimensional diffusion is recurrent if for all x, $\mathbb{P}(\tau_x < \infty) = 1$.*

THEOREM 4.2. *If $a < x < b$ then*

i) *$\mathbb{P}_x(T_a < \infty) = 1$ if and only if $\phi(\infty) = \infty$.*
ii) *$\mathbb{P}_x(T_b < \infty) = 1$ if and only if $\phi(-\infty) = -\infty$.*
iii) *X_t is recurrent if and only if $\phi(\mathbb{R}) = \mathbb{R}$ if and only if both $\phi(\infty) = \infty$ and $\phi(-\infty) = -\infty$.*

PROOF OF THEOREM 4.2. □

5. Intervals with Singular End Points

Let $[\alpha, \beta]$ be an interval such that on any $[r, l] \subset (\alpha, \beta)$ we have that the coefficients $b(x)$ and $\sigma(x)$ are bounded and $\sigma(x)$ positive on $[r, l]$. Under these assumptions the only points were σ can vanish or σ and β become infinite are α and β. Without loss of generality, we assume that $x \in (\alpha, \beta)$.

If we define

$$I_\alpha = \int_\alpha^0 \big[\phi(0) - \phi(z)\big] m(x) dz \qquad I_\beta = \int_0^\beta \big[\phi(z) - \phi(0)\big] m(x) dz$$

$$J_\alpha = \int_\alpha^0 \big[M(0) - M(z)\big] \phi'(x) dz \qquad J_\beta = \int_0^\beta \big[M(z) - M(0)\big] \phi'(x) dz$$

then we have the following result.

THEOREM 5.1. *Let $\gamma \in \{\alpha, \beta\}$, then*

i) *$I_\gamma < \infty$ if and only if X_t can reach the point γ.*
ii) *$J_\gamma < \infty$ if and only if X_t can escape the point γ.*

Following Feller, we have the following boundary point classification.

I_q	J_q	Boundary Type of q
$< \infty$	$< \infty$	regular point
$< \infty$	$= \infty$	absorbing point
$= \infty$	$< \infty$	entrance point
$= \infty$	$= \infty$	natural point

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APPENDIX A

Some Results from Analysis

Recalling that, given a probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable X on such a space we define the expectation of a function f as the integral

$$\mathbb{E}\left[f(X)\right] = \int_\Omega f(\omega)\mathbb{P}(\mathrm{d}\omega)\,,$$

where $\mathbb{P}$ denotes the (probability) measure against which we are integrating. The following results are stated for a general measure μ (*i.e.*, not necessarily a probability measure).

THEOREM 0.1 (Hölder inequality). *Let (Ω, Σ, μ) be a measure space and let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then, for all measurable real- or complex-valued functions f and g on Ω,*

$$\int_\Omega |f(x)g(x)|\mathrm{d}\mu(x) \leqslant \left(\int_\Omega |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q \mathrm{d}\mu(x)\right)^{\frac{1}{q}}\,.$$

THEOREM 0.2 (Lebesgue's Dominated Convergence theorem). *Let $\{f_n\}$ be a sequence of measurable functions on a measure space (Ω, Σ, μ). Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that*

$$|f_n(x)| \leqslant g(x)$$

for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable and

$$\lim_{n\to\infty} \int_\Omega |f_n - f|\, d\mu = 0$$

which also implies

$$\lim_{n\to\infty} \int_\Omega f_n\, d\mu = \int_\Omega f\, d\mu$$

THEOREM 0.3 (Fatou's Lemma). *Given a measure space (Ω, Σ, μ) and a set $X \in \Sigma$, let $\{f_n\}$ be a sequence of $(\Sigma, \mathcal{B}_{\mathbb{R}_{\geqslant 0}})$-measurable non-negative functions $f_n : X \to [0, +\infty]$. Define the function $f : X \to [0, +\infty]$ by setting*

$$f(x) = \liminf_{n\to\infty} f_n(x),$$

for every $x \in X$. Then f is $(\Sigma, \mathcal{B}_{\mathbb{R}_{\geqslant 0}})$-measurable, and

$$\int_\Omega f\, d\mu \leqslant \liminf_{n\to\infty} \int_\Omega f_n\, d\mu.$$

where the integrals may be finite or infinite.

REMARK 0.4. *The above theorem can in particular be used when f is the indicator function $\mathbb{1}_{A_n}$ for a sequence of sets $\{A_n\} \in \Sigma$, obtaining*

$$\mu(\liminf_{n\to\infty} A_n) = \int_\Omega \liminf_{n\to\infty} \mathbb{1}_{A_n}\, d\mu \leqslant \liminf_{n\to\infty} \int_\Omega \mathbb{1}_{A_n}\, d\mu = \liminf_{n\to\infty} \mu(A_n)\,.$$

APPENDIX B

Exponential Martingales and Hermite Polynomials

Let $\sigma(t,\omega)$ be a bounded adapted stochastic process. Define $I(t,\omega) = \int_0^t \sigma_s(\omega)dB(s,\omega)$. We showed that

$$\mathcal{E}_I(t,\omega) := e^{I(t,\omega)-\frac{1}{2}[I](t,\omega)} = \exp\left(\int_0^t \sigma_s(\omega)dB(s\omega) - \frac{1}{2}\int_0^t \sigma_s(\omega)^2 ds\right)$$

was a martingale. $\mathcal{E}(t,\omega)$ is often called the exponential martingale of I. This is reasonable because of the following analogy. In the standard ODE setting we have

$$dY(t) = Y(t)a(t)\,\mathrm{d}t \Longrightarrow Y(t) = Y(0)\exp\left(\int_0^t a(s)ds\right) \ .$$

The analogous SDE is

$$dZ(t,\omega) = Z(t,\omega)dI(t,\omega) = Z(t,\omega)\sigma(t,\omega)dB(t,\omega)$$

or

$$Z(t) = Z(0) + \int_0^t Z(s,\omega)\sigma_s(\omega)dB(s,\omega)$$

The solution to this is $Z(t,\omega) = \mathcal{E}_I(t,\omega)$. Hence it is reasonable to call it the stochastic exponential. From the SDE representation it is clear that $\mathcal{E}_I(t,\omega)$ is a martingale, assuming $I(t,\omega)$ is a nice process (bounded for example). (The Novikov condition is another criteria (in [**8**] or [**17**] for example)).

Just as the exponential can be expanded in a basis of homogeneous polynomials, it is reasonable to ask if something similar can be done with the stochastic exponential. (A function $f(x)$ is homogeneous of degree n if for all $\gamma \in \mathbb{R}$, $f(\gamma x) = \gamma^n f(x)$.) For the regular exponential, we have

$$e^{\gamma X} = \sum_{n=0}^{\infty} \gamma^n \frac{X^n}{n!} \ .$$

Hence we look for $H_n(I,[I])$ such that

$$\mathcal{E}_{\gamma I}(t,\omega) = e^{\gamma I(t,\omega)-\gamma^2\frac{1}{2}[I](t,\omega)} = \sum_{n=0}^{\infty} \gamma^n \frac{H_n\Big(I(t,\omega),[I](t,\omega)\Big)}{n!} \ .$$

Since the stochastic exponential is a martingale, it is reasonable to expect that the $H_n\big(I_t,[I]_t\big)$ should be martingales. In fact, you can argue that the H_n must be mean zero martingales by varying γ. Recall that from its definition $[\gamma I](t,\omega) = \gamma^2[I](t,\omega)$. Hence if we want $H_n\big(\gamma I_t,[\gamma I]_t\big) = \gamma^n H_n\big(I_t,[I]_t\big)$, we are lead to look for polynomials of the form

$$H_n(x,y) = \sum_{0\leqslant m\leqslant\lfloor n/2\rfloor} C_{n,m} x^{n-2m} y^m \ .$$

In homework 2, you found the conditions on the $C_{n,m}$ so that $H_n(I,[I])$ was a martingale. In fact, these polynomial are well known in many areas of math and engineering. They are the Hermite polynomials. They can also defined by the following expression

$$H_n(x,y) = y^n \bar{H}_n\left(\frac{x}{\sqrt{y}}\right)$$
$$\bar{H}_n(z) = (-1)^n e^{\frac{z^2}{2}} \frac{d^n}{dz^n}\left(e^{-\frac{z^2}{2}}\right)$$

Here the $\bar{H}_n$ are the standard Hermite polynomial (possible with a different normalization than you are used to).

We now have two different expressions for the stochastic exponential of $\gamma I(t,\omega)$ with $z(0) = 1$. Namely, setting $Z(t,\omega) = \mathcal{E}_{\gamma I}$, we have

$$Z(t,\omega) = 1 + \gamma \int_0^t Z(s,\omega)\sigma_s(\omega)dB(s,\omega)$$

and

$$Z(t,\omega) = \sum_{k=0}^{\infty} \gamma^n \frac{H_n\Big(I(t,\omega),[I](t,\omega)\Big)}{n!}$$

The first expression has Z on the right hand side. At least formally, we can repeatedly insert the expression of $Z(s,\omega)$. Suppressing the ω dependence, we obtain

$$\begin{aligned}
Z(t) &= 1 + \gamma \int_0^t Z(s_1)\sigma(s_1)dB(s_1) \\
&= 1 + \gamma \int_0^t Z(s_1)\sigma(s_1)dB(s_1) + \gamma^2 \int_0^t \int_0^{s_1} Z(s_2)\sigma(s_2)dB(s_2)\sigma(s_1)dB(s_1) \\
&= 1 + \gamma \int_0^t Z(s_1)\sigma(s_1)dB(s_1) + \cdots \\
&\qquad\qquad + \gamma^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} Z(s_n)\sigma(s_n)dB(s_n)\cdots\sigma(s_1)dB(s_1) \\
&= \sum_{k=0}^{\infty} \gamma^k \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} \sigma(s_k)dB(s_k)\cdots\sigma(s_1)dB(s_1)
\end{aligned}$$

Now if we equate like powers of γ, we obtains

$$H_n\big(I(t),[I](t)\big) = H_n\Big(\int_0^t \sigma dB, \int_0^t \sigma^2 ds\Big) = n! \int_0^t \cdots \int_0^{s_{n-1}} \sigma(s_n)dB(s_n)\cdots\sigma(s_1)dB(s_1)$$

From this expression, it is again clear that $H_n\big(I(t),[I](t)\big)$ is a martingale.

For more information along the lines of this section first see [**12**] and then see [**19, 17**].