MATH 545, Stochastic Calculus
Problem set 5

March 8th, 2019

These problems are due on THU March 21st. You can give them to me in class or drop them in my box. In all of the problems $\mathbb{E}$ denotes the expected value with respect to the specified probability measure $\mathbb{P}$.

**Problem 1.** $X_t$ is a time-homogeneous diffusion with $\mu(x) = 2x$ and $\sigma^2(x) = 4x$. Give its generator $L$. Solve $Lf = 0$, and give a martingale $M_f$ involving $f$. Find the SDE for the process $Y_t = \sqrt{X_t}$, and give the generator of $Y_t$.

By Itô formula we immediately see that the generator $L$ is given by

$$Lf = \left(\mu(x)\partial_x + \frac{1}{2}\sigma^2(x)\partial^2_{xx}\right)f = 2x\left(\partial_x + \partial^2_{xx}\right)f.$$ 

The solution to $Lf = 0$ is given by $f(x) = C_1 + C_2 \exp(x)$ for $C_1, C_2 \in \mathbb{R}$, and consequently $M_f = f(X_t) = \exp[X_t]$.

By Itô formula for $\sqrt{X_t}$ we obtain

$$d\sqrt{X_t} = \frac{1}{2\sqrt{X_t}}dX_t - \frac{1}{2} \frac{1}{4\sqrt{X_t}}d[X]_t = \left(\frac{1}{2\sqrt{X_t}}2X_t - \frac{1}{2} \frac{1}{4\sqrt{X_t}}4X_t\right)dt + \frac{1}{2\sqrt{X_t}}2\sqrt{X_t}dB_t$$

and therefore the SDE for $Y_t = \sqrt{X_t}$ is given by

$$dY_t = \left(Y_t - \frac{1}{2} \frac{1}{Y_t}\right)dt + dB_t.$$

**Problem 2.** Consider the two-dimensional system

$$dX(t) = -Y(t)dt + dB_1(t) - 3dB_2(t)$$
$$dY(t) = X(t)dt - 2dB_1(t) + X(t)dB_2(t)$$

where $B_1$ and $B_2$ are independent one-dimensional Brownian motions. Thus, $B = (B_1, B_2)$ is a two-dimensional Brownian motion. Compute the generator $\mathcal{L}$ for this process, that is, for smooth functions $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$df(X(t), Y(t)) = \mathcal{L}f(X(t), Y(t))dt + \text{something} \cdot dB(t)$$
Hint: recall Itô formula in higher dimensions on p. 118.
As suggested in the hint, the generator is given by the drift term in the (multidimensional) Itô’s formula, which in this case reads (writing $\partial f$ for $\partial f(X(t), Y(t))$)

$$
\begin{align*}
\text{d}f &= \partial_x f \text{d}X(t) + \partial_y f \text{d}Y(t) + \frac{1}{2} \partial_{xx}^2 f \text{d}[X](t) + \frac{1}{2} \partial_{yy}^2 f \text{d}[Y](t) + \partial_{xy} f \text{d}[X, Y](t) \\
&= \partial_x f(-Y(t) \text{d}t + dB_1(t) - 3dB_2(t)) + \partial_y f(X(t) \text{d}t - 2dB_1(t) + X(t)dB_2(t)) + \\
&\quad + \frac{1}{2} \partial_{xx}^2 f(1 + 3^2) \text{d}t + \frac{1}{2} \partial_{yy}^2 f(2^2 + X(t)^2) \text{d}t + \partial_{xy}^2 f(-2 - 3X(t)) \text{d}t.
\end{align*}
$$

From the above we see that the differential operator acting on $f$ in the drift term is

$$
\mathcal{L} f = \left[ -Y(t) \partial_x + X(t) \partial_y + 5 \partial_{xx}^2 + \left( 2 + \frac{X(t)^2}{2} \right) \partial_{yy}^2 - (2 + 3X(t)) \partial_{xy}^2 \right] f.
$$

Problem 3. Let $X_t = X_0 + ct + B_t$, where $B_t$ is a standard Brownian motion starting from zero.

(a). Find the generator $\mathcal{L}$ for this process.
As above, we obtain by Itô formula that

$$
\mathcal{L} f = \left( c \partial_x + \frac{1}{2} \partial_{xx}^2 \right) f.
$$

(b). For given $a < b$, find $f$ that solves $\mathcal{L} f(x) = 0$ with $f(a) = 0$ and $f(b) = 1$.
Hint: The scale function $S$ satisfies $\mathcal{L}S(x) = 0$ (cfr (6.50) on p 162 [Klebaner]), and the two constants in its expression are determined by the boundary conditions.
Recall the scale function solving $\mathcal{L}S(x) = 0$ is given by

$$
S(x) = \int_{x_1}^x \exp \left( - \int_{x_0}^u \frac{2\mu(y)}{\sigma(y)} \text{d}y \right) \text{d}u.
$$

Inserting the values of $\mu(x)$ and $\sigma(x)$ we have the explicit formula

$$
S(x) = \int_{x_1}^x \exp \left( - \int_{x_0}^u 2c \text{d}y \right) \text{d}u = \int_{x_1}^x \exp \left( -2c(u - x_0) \right) \text{d}u \\
= e^{2cx_0} \frac{1}{2c} \left( e^{-2cx_1} - e^{-2cx} \right).
$$

From the above identity we extract the values of $x_0, x_1$ satisfying the boundary conditions, i.e.,

$$
x_1 = a, \quad x_0 = \frac{1}{2c} \log \left( 2c(e^{-2ca} - e^{-2cb}) \right)
$$

(c). Define $\tau = \inf \{ t : X_t \notin (a, b) \}$. Suppose $X_0 = x_0 \in (a, b)$. Find $P(X_\tau = a)$.
By (6.52) in [Klebaner] we have

$$
P_x(X_\tau = a) = \frac{S(b) - S(x)}{S(b) - S(a)} = \frac{e^{-2cx} - e^{-2cb}}{e^{-2ca} - e^{-2cb}} = \frac{e^{2c(b-x)} - 1}{e^{2c(b-a)} - 1}.
$$

We see that $\lim_{x \to a} P(X_\tau = a) = 1$ and $\lim_{x \to b} P(X_\tau = a) = 0$. 

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(d). How does the probability change as $c \to \pm\infty$?

When $x \notin (a, b)$ the above probability converges to 1 for $c \to -\infty$ and to 0 when $c \to \infty$. Otherwise the probability is given in the previous point.