MATH 545, Stochastic Calculus
Problem set 5

March 8th, 2019

These problems are due on THU March 21st. You can give them to me in class or drop them in my box. In all of the problems \( \mathbb{E} \) denotes the expected value with respect to the specified probability measure \( \mathbb{P} \).

**Problem 1.** \( X_t \) is a time-homogeneous diffusion with \( \mu(x) = 2x \) and \( \sigma^2(x) = 4x \). Give its generator \( \mathcal{L} \). Solve \( \mathcal{L} f = 0 \), and give a martingale \( M_f \) involving \( f \). Find the SDE for the process \( Y_t = \sqrt{X_t} \), and give the generator of \( Y_t \).

**Problem 2.** Consider the two-dimensional system

\[
\begin{align*}
    dX(t) &= -Y(t)dt + dB_1(t) - 3dB_2(t) \\
    dY(t) &= X(t)dt - 2dB_1(t) + X(t)dB_2(t)
\end{align*}
\]

where \( B_1 \) and \( B_2 \) are independent one-dimensional Brownian motions. Thus, \( \mathbf{B} = (B_1, B_2) \) is a two-dimensional Brownian motion. Compute the generator \( \mathcal{L} \) for this process, that is, for smooth functions \( f : \mathbb{R}^2 \to \mathbb{R} \), we have

\[
df(X(t), Y(t)) = \mathcal{L} f(X(t), Y(t))dt + \text{something} \cdot dB(t)
\]

Hint: recall Itô’s formula in higher dimensions on p. 118.

**Problem 3.** Let \( X_t = X_0 + ct + B_t \), where \( B_t \) is a standard Brownian motion starting from zero.

(a). Find the generator \( \mathcal{L} \) for this process.

(b). For given \( a < b \), find \( f \) that solves \( \mathcal{L} f(x) = 0 \) with \( f(a) = 0 \) and \( f(b) = 1 \).

Hint: The scale function \( S \) satisfies \( \mathcal{L} S(x) = 0 \) (cfr (6.50) on p 162 [Klebaner]), and the two constants in its expression are determined by the boundary conditions.

(c). Define \( \tau = \inf \{ t : X_t \notin (a, b) \} \). Suppose \( X_0 = x_0 \in (a, b) \). Find \( P(X_{\tau} = a) \).

(d). How does the probability change as \( c \to \pm \infty \)?