These problems are due on TUE Feb 19th. You can give them to me in class, drop them in my box. In all of the problems \( E \) denotes the expected value with respect to the specified probability measure \( P \).

**Problem 0.** Read [Klebaner] Chapter 4 and notes online (by FEB 12th) and [Klebaner] Chapter 5 (by FEB 21st).

**Problem 1** (Klebaner 4.7). A process \( X_t \) has a stochastic differential with \( \mu(x) = bx + c \) and \( \sigma^2(x) = 4x \). Assuming \( X_t \geq 0 \), find the stochastic differential of the process \( Y_t := \sqrt{X_t} \).

By Itô’s formula for \( f(x) = \sqrt{x} \), since \( f'(x) = (2\sqrt{x})^{-1} \) and \( f''(x) = -x^{-3/2}/4 \) we have

\[
d\sqrt{x} = df(X_t) = \frac{1}{2\sqrt{X_t}} dX_t - \frac{1}{2} \frac{X_t^{-3/2}}{4} d[X]_t \\
= \frac{1}{2\sqrt{X_t}} ((bX_t + c) dt + 4X_t dB_t) - \frac{1}{2} \frac{X_t^{-3/2}}{4} (4X_t)^2 dt \\
= \frac{(b - 4)X_t + c}{2\sqrt{X_t}} dt + 2\sqrt{X_t} dB_t.
\]

**Problem 2** (Klebaner 4.8). A process \( X_t \) defined for \( t \in (0,1) \) has a stochastic differential with \( \sigma(x) = x(1-x) \). Assuming \( X_t \in (0,1) \), show that the process \( Y_t := \log(X_t/(1 - X_t)) \) has constant diffusion coefficient.

To find the diffusion coefficient of the process \( Y_t \) we apply Itô’s formula for

\[
f(x) := \log \frac{x}{1-x} \quad \text{with} \quad f'(x) = \frac{1}{x(1-x)}, \quad f''(x) = \frac{1}{x^2(1-x)^2},
\]

and obtain

\[
dY_t = df(X_t) = \frac{1}{X_t(1-X_t)} (\mu(X_t, t) dt + X_t(1-X_t) dB_t) + \frac{1}{2} \frac{1}{X_t^2(1-X_t)^2} \sigma(X_t)^2 dt \\
= \left( \frac{\mu(X_t,t)}{X_t(1-X_t)} + \frac{1}{2} \frac{1}{X_t^2(1-X_t)^2} \sigma(X_t)^2 \right) dt + dB_t,
\]

where we have used the multiplication table for differentials for computing \( d[X]_t \), which only contributes to the \( dt \) term. Hence, we see that the diffusion coefficient for the process is 1 and in particular it is constant.
Again, we apply Itô’s formula to
\[
    f(x) = x^b \quad \text{with} \quad f'(x) = \begin{cases} \frac{b a^{-1}}{\log x} & \text{if } b \neq -1 \\ \\
\end{cases}
\]
and obtain
\[
    dY_t = df(X_t) = bX_t^b - 1(eX_t^a dB_t) + \frac{1}{2} f''(X_t)X_t^2 \cdot dB_t.
\]
where we have again used the multiplication table for differentials. Hence, we see that the above process has a constant differential if \( b = 1 - a \).

Problem 4. Let \( \{ B_t \}_{0 \leq t \leq 1} \) be a 1-dimensional standard Brownian motion on \((\Omega, F, P)\) with the standard filtration \( F = \{ F_t \} \), \( F_t = \sigma(B_s, 0 \leq s \leq t) \). Consider the following simple, adapted process
\[
    X_t(\omega) = 1_{[0,1/2]}(t) + \xi(\omega)1_{(1/2,1)}(t),
\]
where
\[
    \xi(\omega) = \begin{cases} 
5, & \text{if } B_{1/2}(\omega) > 2 \\
0, & \text{if } B_{1/2}(\omega) \leq 2.
\end{cases}
\]
(a) Compute the Itô integral \( I_1(X) = \int_0^1 X_s dB_s \) and its variance.

By the definition above we have
\[
    I_1(X) = \int_0^{1/2} dB_s + \int_{1/2}^1 \xi(\omega) dB_s = B_{1/2} + \int_{1/2}^1 1_{B_{1/2} > 2}(\omega)5 dB_s
\]
Using the independence of the increments \( B_{t+s} - B_t \) of Brownian motion on \( F_t \) and the fact that their expected value is zero, we obtain the variance as
\[
    \text{Var}(I_1(X)) = \text{Var}(B_{1/2}) + \text{Var}(1_{B_{1/2} > 2}(\omega)5(B_1 - B_{1/2}))
\]
where \( \text{Var}(B_{1/2} > 2) \) can be computed integrating the density of a normal distribution with mean 0 and variance 1/2.

(b) Write down the quadratic variation \([I(X)]_t\).
If \( t \leq 1/2 \) we have that
\[
    [I(X)]_t = \left[ \int_0^t 1 dB_s \right] = [B]_t = t.\]
where we have used that the quadratic variation of brownian motion is $t$. When $t > 1/2$ we have

\[
[I(X)]_t = [I(X)]_{0,1/2} + [I(X)]_{1/2, t} = \left[ \int_0^t dB_s \right]_{1/2} + \left[ \int_{1/2}^t \xi(\omega) dB_s \right]_t \\
= [B]_{1/2} + 1_{B_{1/2}}(\omega) \left[ 5 \int_{1/2}^t dB_s \right] = 1/2 + 1_{B_{1/2}}(\omega)25(t - 1/2)
\]

(c). Is $I_1(X)$ Gaussian? Why or why not?

It is a Gaussian Random variable, as its distribution is a sum of Gaussian distributions.

(d). Compute $\mathbb{E}[I_1(X)|\mathcal{F}_{1/4}]$ and $\mathbb{E}[I_1(X)|\mathcal{F}_{3/4}]$.

By the independent increments property of Brownian motion we have that

\[
\mathbb{E}[I_1(X)|\mathcal{F}_{1/4}] = \mathbb{E} \left[ I_{1/4}(X) + \int_{1/4}^1 X_s dB_s | \mathcal{F}_{1/4} \right] = \mathbb{E} \left[ I_{1/4}(X) | \mathcal{F}_{1/4} \right] + \mathbb{E} \left[ \int_{1/4}^1 X_s dB_s | \mathcal{F}_{1/4} \right] \\
= \mathbb{E} \left[ B_{1/4} | \mathcal{F}_{1/4} \right] + \mathbb{E} \left[ (B_{1/2} - B_{1/4}) + 1_{B_{1/2}}(\omega)(B_1 - B_{1/2}) | \mathcal{F}_{1/4} \right] \\
= B_{1/4}(\omega) + 0.
\]

and similarly we have

\[
\mathbb{E}[I_1(X)|\mathcal{F}_{3/4}] = \mathbb{E} \left[ I_{3/4}(X) + \int_{3/4}^1 X_s dB_s | \mathcal{F}_{3/4} \right] = \mathbb{E} \left[ I_{3/4}(X) | \mathcal{F}_{3/4} \right] + \mathbb{E} \left[ \int_{3/4}^1 X_s dB_s | \mathcal{F}_{3/4} \right] \\
= I_{3/4}(X) + 0.
\]

Problem 5 (Part from Klebaner, 4.13, 4.14). Using Itô’s formula, show that the following processes are martingales

(a). $X_t = B_t^3 - 3tB_t$

We define $f(t, x) = x^3 - 3tx$, whose derivatives are given by

\[
\partial_t f(t, x) = 3x, \quad \partial_x f(t, x) = 3x^2 - 3t, \partial_{xx} f(t, x) = 6x.
\]

and by applying the time-dependent version of Itô formula with $dY_t = dB_t (i.e., Y_t = B_t)$ we obtain

\[
dx_t = df(t, Y_t) = \partial_t f(t, Y_t) dt + \partial_x f(t, Y_t) dY_t + \frac{1}{2} \partial_{xx} f(t, Y_t) d[Y]_t \\
= -3B_t dt + (3B_t^2 - 3t) dB_t + \frac{1}{2} 6B_t dt \\
= 0 dt + (3B_t^2 - 3t) dB_t.
\]

From the above we immediately see that the drift coefficient of the process vanishes. Therefore, to show that the process is a martingale we must only check that the Ito integral satisfies the conditions to be a martingale, i.e., we only have to check that

\[
\mathbb{E} \left[ \int_0^t (3B_s^2 - 3s)^2 ds \right] = 9 \int_0^t (\mathbb{E}[B_s^4] - 2\mathbb{E}[B_s^2]s + 3s^2) dt
\]


is finite. This, however, is trivially true as $B_t$ is a gaussian random variable and all its moments are bounded.

(b) $X_t = e^{t/2} \sin(B_t)$ (similarly, $e^{t/2} \cos(B_t)$, but you don’t have to do this).

We define $f(t, x) = e^{t/2} \sin(B_t)$, whose derivatives are given by

$$\partial_t f(t, x) = e^{t/2} \sin(x)/2, \quad \partial_x f(t, x) = e^{t/2} \cos(x), \quad \partial^2_{xx} f(t, x) = -e^{t/2} \sin(B_t).$$

and by applying the time-dependent version of Itô formula with $dY_t = dB_t (i.e., Y_t = B_t)$ we obtain

$$dX_t = df(t, Y_t) = \partial_t f(t, Y_t) \, dt + \partial_x f(t, Y_t) \, dY_t + \frac{1}{2} \partial^2_{xx} f(t, Y_t) \, d[Y]_t$$

$$= e^{t/2} \sin(B_t)/2 \, dt + e^{t/2} \cos(B_t) \, dB_t - \frac{1}{2} e^{t/2} \sin(B_t) \, dt$$

$$= 0 \, dt + e^{t/2} \cos(B_t) \, dB_t \, dB_t.$$

Again, we see that the drift coefficient of the process vanishes, and we check that

$$\mathbb{E} \left[ \int_0^t (e^{s/2} \cos(B_s))^2 \, ds \right] \leq \int_0^t e^s \, ds < \infty.$$

(c) $X_t = (B_t + t)e^{-B_t - \frac{t}{2}}$

We define $f(t, x) = (x + t)e^{-x - \frac{t}{2}}$, whose derivatives are given by

$$\partial_t f(t, x) = -\frac{1}{2}(x+t-2)e^{-x-\frac{t}{2}}, \quad \partial_x f(t, x) = -(x+t-1)e^{-x-\frac{t}{2}}, \quad \partial^2_{xx} f(t, x) = (x+t-2)e^{-x-\frac{t}{2}}.$$

and by applying the time-dependent version of Itô formula with $dY_t = dB_t (i.e., Y_t = B_t)$ we obtain

$$dX_t = df(t, Y_t) = \partial_t f(t, Y_t) \, dt + \partial_x f(t, Y_t) \, dY_t + \frac{1}{2} \partial^2_{xx} f(t, Y_t) \, d[Y]_t$$

$$= -\frac{1}{2} (B_t + t - 2)e^{-B_t - \frac{t}{2}} \, dt - (B_t + t - 1)e^{-B_t - \frac{t}{2}} \, dB_t + \frac{1}{2} (B_t + t - 2)e^{-B_t - \frac{t}{2}} \, dt$$

$$= 0 \, dt - (B_t + t - 1)e^{-B_t - \frac{t}{2}} \, dB_t.$$

Again, we see that the drift coefficient of the process vanishes, and by completing the square and using

that the moments of gaussian random variables are finite we obtain

$$\mathbb{E} \left[ \int_0^t \left( (B_s + s - 1)e^{-B_s - \frac{s}{2}} \right)^2 \, ds \right] < \infty.$$

**Problem 6.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is some polynomial function. Fix $T > 0$ and let

$$\Gamma_n = \left\{ 0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n-1}^{(n)} < t_{k_n}^{(n)} = T \right\}$$

be a sequence of partitions such that $\delta_n = \max_{0 \leq i \leq k_n-1} (t_i^{(n)} - t_{i+1}^{(n)}) \to 0$ as $n \to \infty$. Consider the sums

$$S_n = \sum_{i=0}^{k_n-1} f(B_{t_i^{(n)}})(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})^2, \quad R_n = \sum_{i=0}^{k_n-1} f(B_{t_i^{(n)}})(t_{i+1}^{(n)} - t_i^{(n)}).$$

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(a). What is $E[S_n]$?

We have that

$$E[S_n] = \sum_i E[f(B_{t_i^{(n)}})[B(t_{i+1}^{(n)}) - B(t_i^{(n)})]^2] = \sum_i E[E[f(B_{t_i^{(n)}})|B(t_{i+1}^{(n)}) - B(t_i^{(n)})]^2|\mathcal{F}_{t_i^{(n)}}]]$$

$$= \sum_i E[f(B_{t_i^{(n)}})(t_{i+1}^{(n)} - t_i^{(n)})] = R_n$$

(b). What is $\lim_{n \to \infty} R_n$?

By the definition of Riemann integral we have that we have that

$$\lim_{n \to \infty} R_n = \int_0^t f(B_s) \, ds$$

(c). Show that $\lim_{n \to \infty} E[|S_n - R_n|^2] = 0$.

This is exactly the content of Lemma 4.4 in the notes, so we refer the reader to that result.

**Problem 7.** In this problem, we will show that the Ito integral of a deterministic function is a Gaussian Random Variable. Let $\phi$ be deterministic elementary functions. In other words there exists a sequence of real numbers $\alpha_k : k = 1, 2, \ldots, N$ so that

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty$$

and there exists a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T$$

so that

$$\phi(t) = \sum_{k=1}^{N} \alpha_k 1_{[t_{k-1}, t_k)}(t)$$

(a). Show that if $B_t$ is a standard brownian motion then the Ito integral

$$\int_0^T \phi(t) dB_t$$

is a Gaussian random variable with mean zero and variance

$$\int_0^T \phi(t)^2 dt$$

This is explained in detail in Chapter 4.3 of [Klebaner]

(b). (Optional) Let $f : [0, T] \to \mathbb{R}$ be a deterministic function such that

$$\int_0^T f(t)^2 dt < \infty$$

Then it can be shown that there exists a sequence of deterministic elementary functions $\phi_n$ as above such that

$$\int_0^T (f(t) - \phi_n(t))^2 dt \to 0 \quad \text{as} \quad n \to \infty$$
Assuming this fact, let $\psi_n$ be the characteristic function of the random variable

$$\int_0^T \phi_n(t) dB_t$$

Show that for all $\lambda \in \mathbb{R}$, show that

$$\lim_{n \to \infty} \psi_n(\lambda) = \exp \left( -\frac{\lambda^2}{2} \int_0^T f(t)^2 dt \right)$$

Then use the convergence result on characteristic functions discussed in class to conclude that

$$\int_0^T f(t) dB_t$$

is a Gaussian Random Variable with mean zero and variance

$$\int_0^T f(t)^2 dt$$

by identifying the limit of the characteristic functions above.

**Problem 8 (Optional).** Define $\phi_n(t) = \sin(n\pi t)$ for $n = 1, 2, \ldots$

(a). Check that $\phi_n$’s are orthogonal basis on for the space of functions $L^2([0, 1])$, that is, $\int_0^1 \phi_n(t) \phi_m(t) dt = 0$ for any $n \neq m$. 

(b). Show that $Y_n = I_1(\phi_n) = \int_0^1 \phi_n(t) dB_t$, $n = 1, 2, \ldots$ are uncorrelated Gaussian random variables. You can use Theorem 4.11 in [Klebaner].