MATH 545, Stochastic Calculus
Problem set 3

February 10th, 2019

These problems are due on MON Feb 17th. You can give them to me in class, drop them in my box. In all of the problems $\mathbb{E}$ denotes the expected value with respect to the specified probability measure $\mathbb{P}$.

Problem 0. Read [Klebaner] Chapter 4 (by FEB 17th).

Problem 1 (Klebaner, Problem 3.17)**.** Let $B(t)$ be a Brownian motion. Show that $e^{-\alpha t}B(e^{2\alpha t})$ is a Gaussian process. Find its mean and covariance functions. Does this process have independent increments?

Problem 2. Let $\{B_t\}_{0\le t\le 1}$ be a 1-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with the standard filtration $\mathbb{F} = \{\mathcal{F}_t\}$, $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$. Consider the following simple, adapted process

$$X_t(\omega) = \mathbf{1}_{[0,1/2]}(t) + \xi(\omega)\mathbf{1}_{(1/2,1]}(t),$$

where

$$\xi(\omega) = \begin{cases} 5, & \text{if } B_{1/2}(\omega) > 2 \\ 0, & \text{if } B_{1/2}(\omega) \le 2. \end{cases}$$

Compute the Itô integral $I_1(X) = \int_0^1 X_s dB_s$ and its variance both explicitly and by means of the Ito isometry, and check that the results coincide.

Problem 3. In this problem, we will show that the Ito integral of a deterministic function is a Gaussian Random Variable. Let ϕ be deterministic elementary functions. In other words there exists a sequence of real numbers $\{\alpha_k : k = 1, 2, \dots, N\}$ so that

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty$$

and there exists a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T$$

so that

$$\phi(t) = \sum_{k=1}^{N} \alpha_k \mathbf{1}_{[t_{k-1},t_k)}(t)$$

(a). Show that if B_t is a standard brownian motion then the Ito integral

$$\int_0^T \phi(t) dB_t$$

is a Gaussian random variable with mean zero and variance

$$\int_0^T \phi(t)^2 dt$$

(b). (Optional) Let $f\colon [0,T] \to \mathbf{R}$ be a deterministic function such that

$$\int_0^T f(t)^2 dt < \infty$$

Then it can be shown that there exists a sequence of deterministic elementary functions ϕ_n as above such that

$$\int_0^T (f(t) - \phi_n(t))^2 dt \to 0 \qquad \text{as} \qquad n \to \infty$$

Assuming this fact, let ψ_n be the characteristic function (or Fourier transform) of the random variable

$$\int_0^T \phi_n(t) dB_t$$

Show that for all $\lambda \in \mathbf{R}$, show that

$$\lim_{n\to\infty} \psi_n(\lambda) = \exp\Big(- \frac{\lambda^2}{2}\big(\int_0^T f(t)^2 dt\big)\Big)$$

Then use the the convergence result on characteristic functions discussed in class to conclude that

$$\int_0^T f(t) dB_t$$

is a Gaussian Random Variable with mean zero and variance

$$\int_0^T f(t)^2 dt$$

by identifying the limit of the characteristic functions above.