Problem 0. Read [Klebaner] Chapter 4 and notes online (by FEB 12th) and [Klebaner] Chapter 5 (by FEB 21st).

Problem 1 (Klebaner 4.7). A process $X_t$ has a stochastic differential with $\mu(x) = bx + c$ and $\sigma^2(x) = 4x$. Assuming $X_t \geq 0$, find the stochastic differential of the process $Y_t := \sqrt{X_t}$.

Problem 2 (Klebaner 4.8). A process $X_t$ defined for $t \in (0, 1)$ has a stochastic differential with $\sigma(x) = x(1-x)$. Assuming $X_t \in (0, 1)$, show that the process $Y_t := \log(X_t/(1-X_t))$ has constant diffusion coefficient.

Problem 3 (Klebaner 4.9). A process $X_t$ has a stochastic differential with $\mu(x) = cx$ for $c > 0$ and $\sigma^2(x) = xa$. Let $Y_t := (X_t)^b$. What choice of $b$ will give a constant diffusion coefficient for $Y$?

Problem 4. Let $\{B_t\}_{0 \leq t \leq 1}$ be a 1-dimensional standard Brownian motion on $(\Omega, F, P)$ with the standard filtration $\mathbb{F} = \{\mathcal{F}_t\}, \mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$. Consider the following simple, adapted process

$$X_t(\omega) = 1_{[0,1/2]}(t) + \xi(\omega)1_{(1/2,1]}(t),$$

where

$$\xi(\omega) = \begin{cases} 
5, & \text{if } B_{1/2}(\omega) > 2 \\
0, & \text{if } B_{1/2}(\omega) \leq 2.
\end{cases}$$

(a). Compute the Itô integral $I_1(X) = \int_0^1 X_s dB_s$ and its variance.

(b). Write down the quadratic variation $[I(X)]_t$.

(c). Is $I_1(X)$ Gaussian? Why or why not?

(d). Compute $\mathbb{E}[I_1(X)|\mathcal{F}_{1/4}]$ and $\mathbb{E}[I_1(X)|\mathcal{F}_{3/4}]$.

Problem 5 (Part from Klebaner, 4.13, 4.14). Using Itô’s formula, show that the following processes are martingales.
(a). \( X_t = B_t^3 - 3tB_t \)

(b). \( X_t = e^{t/2} \sin(B_t) \) (similarly, \( e^{t/2} \cos(B_t) \), but you don’t have to do this).

(c). \( X_t = (B_t + t)e^{-B_t - \frac{t^2}{2}} \)

**Problem 6.** Suppose \( f : \mathbb{R} \rightarrow \mathbb{R} \) is some polynomial function. Fix \( T > 0 \) and let

\[
\Gamma_n = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k_n-1}^{(n)} < t_{k_n}^{(n)} = T\}
\]

be a sequence of partitions such that \( \delta_n = \max_{0 \leq i \leq k_n-1} (t_{i+1}^{(n)} - t_i^{(n)}) \rightarrow 0 \) as \( n \rightarrow \infty \). Consider the sums

\[
S_n = \sum_{i=0}^{k_n-1} f(B_{t_i^{(n)}})(B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}})^2, \quad R_n = \sum_{i=0}^{k_n-1} f(B_{t_i^{(n)}})(t_{i+1}^{(n)} - t_{i}^{(n)}).
\]

(a). What is \( \mathbb{E}[S_n] \)?

(b). What is \( \lim_{n \rightarrow \infty} R_n \)?

(c). Show that \( \lim_{n \rightarrow \infty} \mathbb{E}||S_n - R_n||^2 = 0 \).

**Problem 7.** In this problem, we will show that the Ito integral of a deterministic function is a Gaussian Random Variable. Let \( \phi \) be deterministic elementary functions. In other words there exists a sequence of real numbers \( \{\alpha_k : k = 1, 2, \ldots, N\} \) so that

\[
\sum_{k=1}^{\infty} \alpha_k^2 < \infty
\]

and there exists a partition

\[
0 = t_0 < t_1 < t_2 < \cdots < t_N = T
\]

so that

\[
\phi(t) = \sum_{k=1}^{N} \alpha_k \mathbf{1}_{[t_{k-1}, t_k)}(t)
\]

(a). Show that if \( B_t \) is a standard brownian motion then the Ito integral

\[
\int_0^T \phi(t) dB_t
\]

is a Gaussian random variable with mean zero and variance

\[
\int_0^T \phi(t)^2 dt
\]
(b). (Optional) Let \( f : [0, T] \rightarrow \mathbb{R} \) be a deterministic function such that
\[
\int_0^T f(t)^2 \, dt < \infty
\]
Then it can be shown that there exists a sequence of deterministic elementary functions \( \phi_n \) as above such that
\[
\int_0^T (f(t) - \phi_n(t))^2 \, dt \to 0 \quad \text{as} \quad n \to \infty
\]
Assuming this fact, let \( \psi_n \) be the characteristic function of the random variable
\[
\int_0^T \phi_n(t) \, dB_t
\]
Show that for all \( \lambda \in \mathbb{R} \), show that
\[
\lim_{n \to \infty} \psi_n(\lambda) = \exp \left( -\frac{\lambda^2}{2} \left( \int_0^T f(t)^2 \, dt \right) \right)
\]
Then use the convergence result on characteristic functions discussed in class to conclude that
\[
\int_0^T f(t) \, dB_t
\]
is a Gaussian Random Variable with mean zero and variance
\[
\int_0^T f(t)^2 \, dt
\]
by identifying the limit of the characteristic functions above.

**Problem 8** (Optional). Define \( \phi_n(t) = \sin(n\pi t) \) for \( n = 1, 2, \ldots \)

(a). Check that \( \phi_n \)'s are orthogonal basis on for the space of functions \( L^2([0, 1]) \), that is, \( \int_0^1 \phi_n(t) \phi_m(t) \, dt = 0 \) for any \( n \neq m \).

(b). Show that \( Y_n = I_1(\phi_n) = \int_0^1 \phi_n(t) \, dB_t, \) \( n = 1, 2, \ldots \) are uncorrelated Gaussian random variables. You can use Theorem 4.11 in [Klebaner].