MATH 545, Stochastic Calculus
Problem set 2

January 24, 2019

These problems are due on TUE Feb 5th. You can give them to me in class, drop them in my box. In all of the problems $\mathbb{E}$ denotes the expected value with respect to the specified probability measure $\mathbb{P}$.

Problem 0. Read [Klebaner], Chapter4 and Brownian Motion Notes (by FEB 7th)

Problem 1 (Klebaner, Exercise 3.4)**.** Let $\{B_t\}_{t\geq 0}$ be a standard Brownian Motion. Show that, $\{X_t\}_{t\in[0,T]}$, defined as below is a Brownian Motion.

a) $X_t = -B_t\,,$
We check that the defining properties of Brownian motion hold. It is clear that $B_0 = 0$ a.s., and that the increments of the process are independent. For $t > s$, the increments can be written as

$$(-B_t) - (-B_s) = -(B_t - B_s)\,.$$

Because $B_t - B_s$ is a gaussian RV with mean 0 and variance $t - s$, $-(B_t - B_s)$ must have the same properties.

b) $X_t = B_{T-t} - B_T$ for $T < \infty\,,$
It is clear that $B_0 = 0$ a.s.. For $t > s$, the increments of the process are given by

$$X_t - X_s = (B_{T-t} - B_T) - (B_{T-s} - B_T) = B_{T-t} - B_{T-s}\,.$$

These increments are independent of $X_s = B_{T-s} - B_T$ by the inependent increments propery of Brownian motion. The incrememnts are also clearly Gaussian random variables with mean 0 and variance

$$\text{Var}(X_t - X_s) = \text{Var}(B_{T-t} - B_{T-s}) = |T - t - (T - s)| = t - s\,.$$

c) $X_t = cB_{t/c^2}$ for all $c > 0, T < \infty\,,$
The independence of increments and $X_0 = 0$ property are trivial as they are not affected by the scaling. The increments are clearly Gaussian random variables as they are the sum of gaussian random variables, and the scaling preserves the mean 0 property. We The variance of the increments is given by

$$\text{Var}(X_t - X_s) = \text{Var}(cB_{t/c^2} - cB_{s/c^2}) = c^2\text{Var}(B_{t/c^2} - B_{s/c^2}) = c^2(t/c^2 - s/c^2) = t - s\,.$$

d)

$$X_t = \begin{cases} tB_{1/t}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases},[1]$$

By Theorem 3.3 in [Klebaner]: X_t is a mean zero gaussian process with covariance structure $\mathrm{Cov}(X_s, X_t) = \min(s,t)$. Because rescaling time and brownian motion paths does not affect the mean of the process not its Gaussian structure, the first two points above are trivial. Then, for $s < t$ we compute the covariance structure

$$\mathrm{Cov}(X_t, X_s) = \mathrm{Cov}(tB_{1/t}, sB_{1/s}) = ts\mathrm{Cov}(B_{1/t}, B1/s) = \frac{ts}{\min(1/t, 1/s)} = \frac{ts}{t} = \min(t,s)\,.$$

To show continuity of the given process at 0 one could use the strong law of large numbers for a sum of independent gaussian random variables:

$$\frac{1}{n}B_n = \frac{1}{n}\sum_{i=1}^{n}(B_{i+1} - B_i) \to 0$$

in the sense of almost sure convergence of random variables as $N \to \infty$.

Problem 2 (Klebaner, Exercise 3.5)**.** Let B_t and W_t be two independent Brownian Motions. Show that $X_t := (B_t + W_t)/\sqrt{2}$ is also a Brownian Motion. Find the correlation between X_t and B_t.
Clearly $X_0 = 0$ and X_t has independent increments. The increments $X_t - X_s$ are mean 0 Gaussian random variables. The variance of the increments is given by

$$\begin{aligned} \mathrm{Var}(X_t - X_s) &= \frac{1}{2}\mathrm{Var}((B_t + W_t) - (B_s + W_s)) \\ &= \frac{1}{2}\left(\mathrm{Var}(B_t - B_s) + \mathrm{Var}(W_t - W_s) + 2\mathrm{Cov}(B_t - B_s, W_t - W_s)\right) \\ &= \frac{1}{2}\left(t - s + t - s + 0\right) = t - s\,, \end{aligned}$$

where in the second last equality we used that W and B are independent.

Problem 3 (Klebaner, Exercise 3.13)**.** Let $M_t := \max_{0\le s\le t} B_t$. Show that the random variables $|B_t|$, M_t and $M_t - B_t$ have the same distribution for all $t > 0$.
We have seen in class that for $m > 0$ we have $\mathbb{P}\left[M_t > m\right] = 2\mathbb{P}\left[B_t > m\right]$ so

$$\varrho_M(m) = \partial_m \mathbb{P}\left[M_t > m\right] = 2\partial_m \mathbb{P}\left[B_t > m\right] = 2\varrho_{0,0,t}(m)\,.$$

while for $m < 0$ we have $\varrho_M(m) = 0$. Similarly we have that

$$\varrho_{|B|}(m) = \partial_m \mathbb{P}\left[B_t > m\right] + \partial_m \mathbb{P}\left[B_t < -m\right] = 2\varrho_{0,0,t}(m)\,.$$

The third part of the exercise can be solved by simply applying Thm. 3.21 in [Klebaner] and to integrate:

$$\varrho_{M-B}(m) = \int_{-m}^{\infty} \varrho_{B,M}(x, m+x)\,\mathrm{d}x = \int_{-m}^{\infty} \sqrt{\frac{2}{\pi}}\frac{2(m+x) - x}{t^{3/2}} e^{-\frac{(2(m+x)-x)^2}{2t}}\,\mathrm{d}x$$

[1]Hint: Use Theorem 3.3 in [Klebaner]. Some extra work is needed at $t = 0$ here to prove continuity. To this end, you can use that $\lim_{t\to 0} tB_{1/t} = \lim_{n\to\infty} n^{-1}B_n$.

$$= \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{m+x}{t^{3/2}} e^{-\frac{(m+x)^2}{2t}} \,\mathrm{d}x = 2\partial_m \left[\frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-\frac{(m+x)^2}{2t}} \,\mathrm{d}x \right]$$
$$= \partial_m \left[\frac{1}{\sqrt{2\pi t}} \int_m^\infty e^{-\frac{x^2}{2t}} \,\mathrm{d}x \right] = 2\varrho_{0,0,t}(m)\,.$$

for $m > 0$, where in the second and third line we have performed a change of variables.

Problem 4. Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion.

a) For any $0 \leq s < t$, show that the joint distribution of (B_s, B_t) is a bivariate normal distribution and determine the mean vector μ and covariance matrix Σ of this bivariate normal distribution. [2]
We use the result at the end of pp. 59 of [Klebaner] for the random variables $X = B_s$ and $Y = B_t - B_s$, which are independent gaussian random variables with mean 0 and variances s and $s - t$ respectivaly. Consequently, the random vector $(B_s, B_t) = (X, X + Y)$ is distributed according to a 2-dimensional gaussian distribution with mean vector $\mu = (0, 0)$ and covariance matrix

$$\sigma_0 = \begin{pmatrix} s & s \\ s & t \end{pmatrix}.$$

b) Find a matrix $A = \begin{bmatrix} a_{ss} & a_{st} \\ a_{ts} & a_{tt} \end{bmatrix}$ so that $[Z_1\ Z_2]$, defined as follows, has a standard bivariate normal distribution.

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} := A \begin{bmatrix} B_s \\ B_t \end{bmatrix} = \begin{bmatrix} a_{ss} & a_{st} \\ a_{ts} & a_{tt} \end{bmatrix} \begin{bmatrix} B_s \\ B_t \end{bmatrix} = \begin{bmatrix} a_{ss}B_s + a_{st}B_t \\ a_{ts}B_s + a_{tt}B_t \end{bmatrix}$$

A standard bivariate normal distribution is a bivariate normal distribution where the means of both coordinate variables are zero and the covariance matrix is the identity matrix. You can use the fact that any linear combination of random variables following a multi-variate normal distribution has a normal distribution.
Let $[Z_1\ \ Z_2]$ have a standard bivariate normal distribution. We have

$$\begin{bmatrix} B_s \\ B_t \end{bmatrix} \overset{D}{=} \begin{bmatrix} \sqrt{s} & 0 \\ \sqrt{s} & \sqrt{t-s} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

To see this, one can check that the right side has a centered bivariate normal distribution with covariance matrix $\begin{bmatrix} s & s \\ s & t \end{bmatrix}$. Thus, A must be the inverse of $\begin{bmatrix} \sqrt{s} & 0 \\ \sqrt{s} & \sqrt{t-s} \end{bmatrix}$, i.e.,

$$A = \begin{bmatrix} \sqrt{s} & 0 \\ \sqrt{s} & \sqrt{t-s} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{s}} & 0 \\ -\frac{1}{\sqrt{t-s}} & \frac{1}{\sqrt{t-s}} \end{bmatrix}.$$

[2] Hint: The discussion at the end of pp. 59 of [Klebaner] should be useful. Or using the independence of B_s and $B_t - B_s$, you can first find the joint density of

$$(B_s,\ B_t - B_s).$$

Then do a transformation to get the joint density of (B_s, B_t) and recognize that it is a density of some bivariate normal distribution.

Problem 5 (More martingales in Brownian Motion)**.** Let $\{B_t\}_{t\geq 0}$ be a standard Brownian Motion with a natural filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$.

(a) Compute $\mathbb{E}\left[B_t^4|\mathcal{F}_s\right]$ for $t > s \geq 0$.
We have that

$$\begin{aligned}
\mathbb{E}\left[B_t^4|\mathcal{F}_s\right] &= \mathbb{E}\left[(B_s + (B_t - B_s))^4|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[B_s^4 + 4B_s^3(B_t - B_s) + 4B_s^2(B_t - B_s)^2 + 4B_s(B_t - B_s)^3 + (B_t - B_s)^4|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[B_s^4|\mathcal{F}_s\right] + \mathbb{E}\left[4B_s^3(B_t - B_s)|\mathcal{F}_s\right] + \mathbb{E}\left[6B_s^2(B_t - B_s)^2|\mathcal{F}_s\right] \\
&\qquad + \mathbb{E}\left[4B_s(B_t - B_s)^3|\mathcal{F}_s\right] + \mathbb{E}\left[(B_t - B_s)^4|\mathcal{F}_s\right] \\
&= B_s^4 + 4B_s\mathbb{E}\left[(B_t - B_s)^3|\mathcal{F}_s\right] + B_s^2\mathbb{E}\left[6(B_t - B_s)^2|\mathcal{F}_s\right] \\
&\qquad + 4B_s\mathbb{E}\left[(B_t - B_s)^3|\mathcal{F}_s\right] + \mathbb{E}\left[(B_t - B_s)^4|\mathcal{F}_s\right] \\
&= B_s^4 + 6B_s^2\mathbb{E}\left[(B_t - B_s)^2|\mathcal{F}_s\right] + \mathbb{E}\left[(B_t - B_s)^4|\mathcal{F}_s\right] \\
&= B_s^4 + 6B_s^2(t - s) + 3(t - s)^2 .
\end{aligned}$$

where in the fifth equality we have used that oddmoments of normal distributions are 0 and in the last the definition of brownian motion increments.

(b) Consider function $f_4(t, x) = x^4 - 6tx^2 + 3t^2$. Show that $\{M_t\}_{t\geq 0}$ given by

$$M_t = f_4(t, B_t) = B_t^4 - 6tB_t^2 + 3t^2$$

is a martingale adapted to the filtration $\mathbb{F}$.
We only check that $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$. In light of the above we have that

$$\begin{aligned}
\mathbb{E}\left[M_t|\mathcal{F}_s\right] &= \mathbb{E}\left[B_t^4|\mathcal{F}_s\right] - 6t\mathbb{E}\left[B_t^2|\mathcal{F}_s\right] + 3t^2 = \mathbb{E}\left[B_t^4|\mathcal{F}_s\right] - 6t(\mathbb{E}\left[B_s^2|\mathcal{F}_s\right] + (t - s)) + 3t^2 \\
&= B_s^4 + 6B_s^2(t - s) + 3(t - s)^2 - 6t(B_s^2 + (t - s)) + 3t^2 = B_s^4 - 6sB_s^2 + 3s^2
\end{aligned}$$

(c) We have shown in class that $M_t = f(t, B_t)$ is a martingale for the cases

$$\begin{aligned}
f(t, x) &= f_1(t, x) := x; \\
f(t, x) &= f_2(t, x) := x^2 - t; \quad \text{and} \\
f(t, x) &= g(\vartheta, t, x) := e^{\vartheta x - \vartheta^2 t/2},
\end{aligned}$$

and the corresponding martingales are $\{B_t\}_{t\geq 0}$, $\{B_t^2 - t\}$ and $\left\{e^{\vartheta B_t - \vartheta^2 t/2}\right\}_{t\geq 0}$. Check that f_1, f_2 and f_4 are solutions of the following (time reversed) heat equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) f(t, x) = 0 \qquad \text{(PDE)}$$

with initial condition $f_n(0, x) = x^n$, for $n = 1, 2, 4$.
The proof is immediate by inserting the suggested solution in the left hand side of the PDE and checking that the result is 0.

(d) (Optional) Find $f_3(t,x)$ so that (i) it is a solution to (PDE) with initial condition $f_3(0,x) = x^3$ and (ii) $\{f_3(t,B_t)\}_{t\geq 0}$ is a martingale. Hint: You can do this part by solving the PDE from the given initial condition (if you know how to), or by computing $\mathbb{E}\left[B_t^3|\mathcal{F}_s\right]$ and guessing what $f_3(t,x)$ should look like.

(e) (Optional). Check that, in fact, we have

$$f_1(t,x) = \left.\frac{\partial g(\vartheta,t,x)}{\partial \vartheta}\right|_{\vartheta=0}$$
$$f_2(t,x) = \left.\frac{\partial^2 g(\vartheta,t,x)}{\partial \vartheta^2}\right|_{\vartheta=0}, \qquad \text{etc.}$$

You can use this part to find your solution to part (d).

Problem 6 (Klebaner, Exercise 3.14)**.** The first zero of the Standard Brownian Motion is $B_0 = 0$. What is the second zero?
By the arcsine law seen in class, the probability that Brownian motion has a zero in the interval (a,b) is given by

$$\mathbb{P}\left[B_t = 0 \text{ for } t \in (a,b)\right] = \frac{2}{\pi}\arcsin\frac{a}{b}\,.$$

We see that setting $a = 0$ and taking $b \to 0$ results ina probability of 1, so there the second 0 of brownian motion is also at 0.

An alternative way to prove the same result is by arguing similarly to Example 3.8 in [Klebaner]: it is possible that the probability that the sign of brownian motion is constant in an $(0,\varepsilon)$ is 0 independently of ε, proving the result.

Problem 7 (Optional: Diffusion and Brownian Motion)**.** Let B_t be a standard Brownian Motion starting from zero and define

$$p(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$$

Given any $x \in \mathbb{R}$, define $X_t = x + B_t$. Of course X_t is just a Brownian Motion stating from x at time 0. Fixing a smooth bounded function $\mathbb{R} \to \mathbb{R}$, we define the function $u(x,t)$ by

$$u(x,t) = \mathbb{E}_x[f(X_t)]$$

where we have decorated the expectation with the subscript x to remind us that we are starting from the point x.

- Explain why
$$u(x,t) = \int_{\infty}^{\infty} f(y)p(t,x-y)dy$$
- Show by direct calculation using the formula from the previous question that for $t > 0$, $u(x,t)$ satisfies the diffusion equation
$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}$$
for some constant c. (Find the correct c !)

- Show that
$$\lim_{t\to 0} u(t,x) = f(x)$$
and hence the initial condition for the diffusion equation is f.

Problem 8 (Optional)**.** Let $\{\eta_k : k = 0, \cdots\}$ be a collection of mutually independent standard Gaussian random variable with mean zero and variance one. Define

$$X(t) = \frac{t}{\sqrt{\pi}}\eta_0 + \sqrt{\frac{2}{\pi}}\sum_{k=1}^{\infty}\frac{\sin(kt)}{k}\eta_k \ .$$

Show that on the interval $[0,\pi]$ $X(t)$ has the same mean, variance and covariance as Brownian Motion.

In fact, it is Brownian Motion. For extra credit, prove this. (There are a number of ways to do this. One is to see X as the limit of the finite sums which are each continuous functions. Then prove that X is the uniform limit of these continuous functions and hence is itself continuous.)

Then observe that "formally" the time derivative of $X(t)$ is the sum of all frequencies with a random amplitudes which are independent and identical $N(0,1)$ Gaussian random variables. This is the origin of the term "white noise" since all frequencies are equally represented as in white light.

In the above calculations you may need the fact that

$$\min(t,s) = \frac{ts}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(kt)\sin(ks)}{k^2} \ .$$

(If you are interested, this can be shown by periodically extending $\min(t,s)$ to the interval $[-\pi,\pi]$ and then showing that it has the same Fourier transform as the right-hand side of the above expression. Then use the fact that two continuous functions with the same Fourier transform are equal on $[-\pi,\pi]$.)

Problem 9 (optional)**.** If S and T are both stopping times relative to the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$, then $\max(S,T)$ and $S+T$ are also stopping times relative to $\mathbb{F}$.

We need to check that for any $t \geq 0$, $\{S+T \leq t\} \in \mathcal{F}_t$, which is equivalent to $\{S+T > t\} \in \mathcal{F}_t$. To do this, we write

$$\{S+T>t\} = \{S=0, S+T>t\} \cup \{S=t, S+T>t\} \cup \{0<S<t, S+T>t\}$$

Apparently,

$$\{S=0, S+T>t\} = \{S=0\} \cap \{T>t\} \in \mathcal{F}_t$$
$$\{S=t, S+T>t\} = \{S=t\} \cap \{T>0\} \in \mathcal{F}_t$$

So it suffices to show the third set $\{t > S > 0, S+T > t\} \in \mathcal{F}_t$. Notice

$$\{0<S<t, S+T>t\} = \bigcup_{r\in\mathbb{Q}\cap(0,t)} \{r<S<t,\ T>t-r\} \qquad (**)$$

The right side of $(**)$ is in $\mathcal{F}_t$, because it is a countable union of sets in $\mathcal{F}_t$: for each $r \in \mathbb{Q}\cap(0,t)$

$$\{r<S<t,\ T>t-r\} = \{r<S<t\} \cap \{T>t-r\} \in \mathcal{F}_t$$

To show $(**)$, denote the left side and the right side by A_L and A_R, respectively. For each $r \in \mathbb{Q} \cap (0,t)$,

$$\begin{aligned}\{r < S(\omega) < t, T(\omega) > t - r\} &\subset \{r < S(\omega) < t, T(\omega) + S(\omega) > t\} \\ &\subset \{0 < S(\omega) < t, T(\omega) + S(\omega) > t\} = A_L\end{aligned}$$

which means $A_R \subset A_L$. Moreover, for any $\omega \in A_L$, choose

$$0 < \varepsilon < \min\{S(\omega) + T(\omega) - t, S(\omega)\}, \text{ and then choose } q \in (S(\omega) - \varepsilon, S(\omega) - \varepsilon/2) \cap \mathbb{Q}.$$

Then we have $q < S(\omega) < t$ and

$$S(\omega) + T(\omega) > t + \varepsilon \quad \Rightarrow \quad T(\omega) > t - (S(\omega) - \varepsilon) > t - q$$

thus $\omega \in \{q < S < t, T > t - q\}$ for such choice of $q \in \mathbb{Q}$. This means for any $\omega \in A_L$, we can find q such that $\omega \in \{q < S < t, T > t - q\} \subset A_R$ and hence $A_L \subset A_R$.