MATH 545, Stochastic Calculus
Problem set 2

January 29th, 2019

These problems are due on MON Feb 10th. You can give them to me in class or drop them in my box. In all of the problems $\mathbb{E}$ denotes the expected value with respect to the specified probability measure $\mathbb{P}$.

Problem 0. Read [Klebaner], Chapter 4 and Brownian Motion Notes (by FEB 10th)

Problem 1 (Klebaner, Exercise 3.4)**.** Let $\{B_t\}_{t\geq 0}$ be a standard Brownian Motion. Show that, $\{X_t\}_{t\in[0,T]}$, defined as below is a Brownian Motion.

a) $X_t = -B_t$,

b) $X_t = B_{T-t} - B_T$ for $T < \infty$,

c) $X_t = cB_{t/c^2}$ for all $c > 0, T < \infty$,

d)
$$X_t = \begin{cases} tB_{1/t}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases} ,[1]$$

Problem 2 (Klebaner, Exercise 3.5)**.** Let B_t and W_t be two independent Brownian Motions. Show that $X_t := (B_t + W_t)/\sqrt{2}$ is also a Brownian Motion. Find the correlation between X_t and B_t .

Problem 3 (Klebaner, Exercise 3.13)**.** Let $M_t := \max_{0\leq s\leq t} B_t$. Show that the random variables $|B_t|$, M_t and $M_t - B_t$ have the same distribution for all $t > 0$.

Problem 4. Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion.

a) For any $0 \leq s < t$, show that the joint distribution of (B_s, B_t) is a bivariate normal distribution and determine the mean vector μ and covariance matrix Σ of this bivariate normal distribution. [2]

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[1] Hint: Use Theorem 3.3 in [Klebaner]. A little extra work is needed at $t = 0$ here to prove continuity. To this end, you can use that $\lim_{t\to 0} tB_{1/t} = \lim_{n\to\infty} n^{-1}B_n$.

[2] Hint: The discussion at the end of pp. 59 of [Klebaner] should be useful. Or using the independence of B_s and $B_t - B_s$, you can first find the joint density of

$$(B_s,\ B_t - B_s).$$

Then do a transformation to get the joint density of (B_s, B_t) and recognize that it is a density of some bivariate normal distribution.

b) Find a matrix $A = \begin{bmatrix} a_{ss} & a_{st} \\ a_{ts} & a_{tt} \end{bmatrix}$ so that $[Z_1\ Z_2]$, defined as follows, has a standard bivariate normal distribution.

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} := A \begin{bmatrix} B_s \\ B_t \end{bmatrix} = \begin{bmatrix} a_{ss} & a_{st} \\ a_{ts} & a_{tt} \end{bmatrix} \begin{bmatrix} B_s \\ B_t \end{bmatrix} = \begin{bmatrix} a_{ss}B_s + a_{st}B_t \\ a_{ts}B_s + a_{tt}B_t \end{bmatrix}$$

A standard bivariate normal distribution is a bivariate normal distribution where the means of both coordinate variables are zero and the covariance matrix is the identity matrix. You can use the fact that any linear combination of random variables following a multi-variate normal distribution has a normal distribution.

Problem 5 (More martingales in Brownian Motion)**.** Let $\{B_t\}_{t\geq 0}$ be a standard Brownian Motion with a natural filtration $\mathfrak{F} = \{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$.

(a) Compute $\mathbb{E}\left[B_t^4 | \mathcal{F}_s\right]$ for $t > s \geq 0$.

(b) Consider function $f_4(t,x) = x^4 - 6tx^2 + 3t^2$. Show that $\{M_t\}_{t\geq 0}$ given by

$$M_t = f_4(t, B_t) = B_t^4 - 6tB_t^2 + 3t^2$$

is a martingale adapted to the filtration $\mathbb{F}$.

(c) We have shown in class that $M_t = f(t, B_t)$ is a martingale for the cases

$$\begin{aligned} f(t,x) &= f_1(t,x) := x; \\ f(t,x) &= f_2(t,x) := x^2 - t; \quad \text{and} \\ f(t,x) &= g(\vartheta, t, x) := e^{\vartheta x - \vartheta^2 t/2}, \end{aligned}$$

and the corresponding martingales are $\{B_t\}_{t\geq 0}$, $\{B_t^2 - t\}$ and $\left\{e^{\vartheta B_t - \vartheta^2 t/2}\right\}_{t\geq 0}$. Check that f_1, f_2 and f_4 are solutions of the following (time reversed) heat equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) f(t,x) = 0 \qquad \text{(PDE)}$$

with initial condition $f_n(0,x) = x^n$, for $n = 1, 2, 4$.

(d) (Optional) Find $f_3(t,x)$ so that (i) it is a solution to (PDE) with initial condition $f_3(0,x) = x^3$ and (ii) $\{f_3(t, B_t)\}_{t\geq 0}$ is a martingale. Hint: You can do this part by solving the PDE from the given initial condition (if you know how to), or by computing $\mathbb{E}\left[B_t^3 | \mathcal{F}_s\right]$ and guessing what $f_3(t,x)$ should look like.

(e) (Optional). Check that, in fact, we have

$$\begin{aligned} f_1(t,x) &= \left.\frac{\partial g(\vartheta, t, x)}{\partial \vartheta}\right|_{\vartheta=0} \\ f_2(t,x) &= \left.\frac{\partial^2 g(\vartheta, t, x)}{\partial \vartheta^2}\right|_{\vartheta=0}, \quad \text{etc.} \end{aligned}$$

You can use this part to find your solution to part (d).

Problem 6 (Klebaner, Exercise 3.14)**.** The first zero of the Standard Brownian Motion is $B_0 = 0$. What is the second zero?

Problem 7 (Optional: Diffusion and Brownian Motion)**.** Let B_t be a standard Brownian Motion starting from zero and define

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

Given any $x \in \mathbb{R}$, define $X_t = x + B_t$. Of course X_t is just a Brownian Motion stating from x at time 0. Fixing a smooth bounded function $\mathbb{R} \to \mathbb{R}$, we define the function $u(x,t)$ by

$$u(x,t) = \mathbb{E}_x[f(X_t)]$$

where we have decorated the expectation with the subscript x to remind us that we are starting from the point x.

- Explain why
$$u(x,t) = \int_{\infty}^{\infty} f(y) p(t, x-y) dy$$
- Show by direct calculation using the formula from the previous question that for $t > 0$, $u(x,t)$ satisfies the diffusion equation
$$\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2}$$
for some constant c. (Find the correct c !)
- Show that
$$\lim_{t \to 0} u(t,x) = f(x)$$
and hence the initial condition for the diffusion equation is f.

Problem 8 (Optional)**.** Let $\{\eta_k : k = 0, \cdots\}$ be a collection of mutually independent standard Gaussian random variable with mean zero and variance one. Define

$$X(t) = \frac{t}{\sqrt{\pi}} \eta_0 + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{\sin(kt)}{k} \eta_k \ .$$

Show that on the interval $[0, \pi]$ $X(t)$ has the same mean, variance and covariance as Brownian Motion.

In fact, it is Brownian Motion. For extra credit, prove this. (There are a number of ways to do this. One is to see X as the limit of the finite sums which are each continuous functions. Then prove that X is the uniform limit of these continuous functions and hence is itself continuous.)

Then observe that "formally" the time derivative of $X(t)$ is the sum of all frequencies with a random amplitudes which are independent and identical $N(0,1)$ Gaussian random variables. This is the origin of the term "white noise" since all frequencies are equally represented as in white light.

In the above calculations you may need the fact that

$$\min(t,s) = \frac{ts}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(kt)\sin(ks)}{k^2} \ .$$

(If you are interested, this can be shown by periodically extending $\min(t,s)$ to the interval $[-\pi, \pi]$ and then showing that it has the same Fourier transform as the right-hand side of the above expression. Then use the fact that two continuous functions with the same Fourier transform are equal on $[-\pi, \pi]$.)

Problem 9 (Optional)**.** Show that if S and T are both stopping times relative to the filtration $\mathfrak{F} = \{\mathcal{F}_t\}_{t\geq 0}$, then $\max(S, T)$ and $S + T$ are also stopping times relative to $\mathfrak{F}$.