MATH 545, Stochastic Calculus
Problem set 1

January 27,2006

These problems are due on THU Jan 24. You can give them to me in class, drop them in my box. In all of the problems $\mathbb{E}$ denotes the expected value with respect to the specified probability measure $\mathbb{P}$.

Problem 0. Read [Klebaner], Chapter 2 (by JAN 22), Chapter 3 (by JAN 29) (terminology: σ-field = σ-algebra)

Problem 1. Consider the following probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \big\{(\omega_0, \omega_1) : \omega_i \in \{-1, 0, 1\}\big\}$. Take of each of the ω_i to be mutually independent with $\mathbb{P}[\omega_i = 0] = 1/2$ and $\mathbb{P}[\omega_i = \pm 1] = 1/4$. ($\mathcal{F}$ is just the σ-algebra generated by the collection of single points, but this is not important).

For $n = 0$ or 1 define the random variables X_n by $X_0 = \omega_0$ and $X_1 = \omega_0 \omega_1$.

a) What is $\mathbb{P}[X_1 = 1]$? What is $\mathbb{E}[X_0]$?

Clearly $\mathbb{E}[X_0] = 0$ and $\mathbb{P}[X_1 = 1] = \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{8}$.

b) Let A be the event that $\{X_1 \neq 0\}$. What is $\sigma(A)$?

It is the σ-algebra generated by $\{\{(1,0),(-1,0),(0,0),(0,1),(0,-1)\},\{(1,1),(-1,-1),(-1,1),(1,-1)\}\}$

c) What is $\sigma(X_0)$ and $\sigma(X_1)$?

The σ-algebra generated by X_1 is the one generated by

$$\{\{(1,0),(-1,0),(0,0),(0,1),(0,-1)\},\{(1,1),(-1,-1)\},\{(-1,1),(1,-1)\}\}$$

and the one generated by X_0 is generated by

$$\{\{(0,0),(0,1),(0,-1)\},\{(1,0),(1,-1),(1,1)\},\{(-1,1),(-1,-1),(-1,0)\}\}.$$

d) What is $\mathbb{E}[X_1|A]$? What is $\mathbb{E}[X_1|X_0]$?

It suffices to average over the three different sets generating the σ-algebra: by symmetry $\mathbb{E}[X_1|A] = \mathbb{E}[X_1|X_0] = 0$.

Problem 2. Consider the sample space $\Omega = \{\omega_i\}_{i=1}^3$ with ω_i are independent identically distributed according to

$$\omega_i = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

Furthermore, define $X_n(\omega) = \sum_{i=1}^n \omega_i$ and let $\mathcal{F}_1 = \sigma(X_1), \mathcal{F}_{12} = \sigma(\{\omega_1, \omega_2\})$ (the sigma algebras generated by the outcome of the first flip and of the first two flips respectively). Compute

a) $\mathbb{E}[X_3|X_1]$

Similarly to what was done in class we can write:

$$\mathbb{E}[X_3|X_1] = \mathbb{E}[X_1 + \omega_2 + \omega_3|X_1] = \mathbb{E}[X_1|X_1] + \mathbb{E}[\omega_2 + \omega_3|X_1] = X_1 + \mathbb{E}[\omega_2 + \omega_3] = X_1$$

b) $\mathbb{E}[\mathbb{E}[X_3|\mathcal{F}_1]|\mathcal{F}_{12}]$

Since $\mathcal{F}_1 \subseteq \mathcal{F}_{12}$ we have

$$\mathbb{E}[\mathbb{E}[X_3|\mathcal{F}_1]|\mathcal{F}_{12}] = \mathbb{E}[X_3|\mathcal{F}_1] = \mathbb{E}[X_3|X_1] = X_1$$

c) $\mathbb{E}[\mathbb{E}[X_3|\mathcal{F}_{12}]|\mathcal{F}_1]$

In this case we have that $\mathbb{E}[X_3|\mathcal{F}_{12}] = \mathbb{E}[X_2 + \omega_3|\mathcal{F}_{12}] = \mathbb{E}[X_2|\mathcal{F}_{12}] + \mathbb{E}[\omega_3|\mathcal{F}_{12}] = X_2$ so that

$$\mathbb{E}[\mathbb{E}[X_3|\mathcal{F}_{12}]|\mathcal{F}_1] = \mathbb{E}[X_2|\mathcal{F}_1] = \mathbb{E}[X_1|X_1] + \mathbb{E}[\omega_2|X_1] = X_1$$

d) check that the answer to the above two points is the same (tower property)

Trivial

Suggestion: write a random variable Y (conditional expectations are random variables!) on σ-algebra $\mathcal{F}$ with a countable number of elements as

$$Y(\omega) = \sum_k 1_{A_k}(\omega) a_k \qquad \text{for some } A_k \in \mathcal{F}, a_k \in \mathbb{R}\,,$$

where $1_A(\omega)$ is the indicator function on the set A.

Problem 3. For a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$, prove the Chebychev inequality:

$$\mathbb{P}[|X| \geq \lambda] \leq \frac{\mathbb{E}[|X|^p]}{\lambda^p}$$

for all $1 \leq p < \infty$ as well as its exponential form

$$\mathbb{P}[|X| \geq \lambda] \leq \frac{\mathbb{E}\left[e^{k|X|}\right]}{e^{k\lambda}}$$

when the integrals on the right are finite. Here k and λ are positive constants. (Hint: denoting by $\mathbb{1}\{A\}$ the indicator function on the set A combine the facts that $\mathbb{P}[|X| \geq \lambda] = \mathbb{E}[\mathbb{1}\{|X| \geq \lambda\}]$, $\mathbb{1}\{|X| \geq \lambda\}\lambda^p \leq \mathbb{1}\{|X| \geq \lambda\}|X|^p$, and $1 = \mathbb{1}\{|X| \geq \lambda\} + \mathbb{1}\{|X| < \lambda\}$.)

We prove the first of the two inequalities, the second is proven following the same steps. We note that

$$X(\omega)^p \geq \lambda^p \mathbb{1}_{X \geq \lambda}(\omega)$$

for any $p > 0$ and therefore we have

$$\mathbb{E}[X^p] \geq \mathbb{E}[\lambda^p \mathbb{1}_{X \geq \lambda}] = \lambda^p \mathbb{E}[\mathbb{1}_{X \geq \lambda}] = \lambda^p \mathbb{P}[X \geq \lambda]\ .$$

Problem 4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probabilty space and let A_k, $k = 1, 2, 3, \ldots$ be a sequence of events in $\mathcal{F}$ for which

$$\sum_{k=1}^{\infty} \mathbb{P}\left[A_k\right] < \infty$$

Prove the (first) Borell-Cantelli lemma which says that

$$\mathbb{P}\left[\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} A_k\right] = 0$$

Interpret this event whose probability is zero. (Say something like "it is the event in which all of the events A_k happen". This of course not the right answer.)

There are different proofs of this lemma. We report one below.

Let $\{A_n\}$ be a sequence of events in some probability space and suppose that the sum of the probabilities of the events A_n is finite. That is suppose:

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty.$$

As the series converges we must have that

$$\sum_{n=N}^{\infty} \mathbb{P}(A_n) \to 0, \quad \text{as } N \to \infty.$$

Therefore :

$$\lim_{N \to \infty} \sum_{n=N}^{\infty} \mathbb{P}(A_n) = 0.$$

Combining the above results with the subadditivity property of $\mathbb{P}$, *i.e.*, that for all sequences $\{A_n\}$ of sets (not necessarily disjoint) we have

$$\mathbb{P}\left(\bigcup_{n=N}^{\infty} A_n\right) \leq \sum_{n=N}^{\infty} \mathbb{P}(A_n),$$

we obtain

$$\mathbb{P}\left(\limsup_{n \to \infty} A_n\right) = \mathbb{P}(\text{infinitely many of the } A_n \text{ occur}) = \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n\right)$$

$$= \lim_{N \to \infty} \mathbb{P}\left(\bigcup_{n=N}^{\infty} A_n\right) \leqslant \lim_{N \to \infty} \sum_{n=N}^{\infty} \mathbb{P}(A_n) = 0.$$

Now let us use this result: Let X_k be a sequence of random variables and suppose that there is another random variable X such that if

$$A_k = \{\omega \in \Omega \mid \ |X_k(\omega) - X(\omega)| > \delta\}$$

then

$$\sum_{k=1}^{\infty} \mathbb{P}\left[A_k\right] < \infty$$

for all $\delta > 0$. Show that $X_n \to X$ with probability one. (That is the set of ω so that $X_n(\omega) \to X(\omega)$ has measure (probability) one.

Basically, the first Borel-Cantelli lemma is the principle means by which information about expectations can be converted into almost sure information.

Here, we have convergence if for any δ, the event $\omega \in A_k$ happens only finitely many times. By assumption, however, we have $\sum_{k=1}^{\infty} \mathbb{P}[A_k] < \infty$ which implies, by Borel Cantelli, that such event has probability 1.

More precisely by definition of limit, $\omega \in \{\lim_n X_n = X\}$ if and only if for all $m \geq 1$ there exists $n \geq 1$ such that for every $i \geq n$ it holds $|X_i(\omega) - X(\omega)| \leq 1/m$. This is true if and only if

$$\omega \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (A_i(1/m))^c = \left(\bigcup_{m=1}^{\infty} \limsup_n A_n(1/m) \right)^c.$$

Then

$$\mathbb{P}(\lim_n X_n = X) = 1 - \mathbb{P}\left(\bigcup_{m=1}^{\infty} \limsup_n A_n(1/m) \right) \geq 1 - \sum_{m=1}^{\infty} \mathbb{P}(\limsup_n A_n(1/m)) = 1.$$

Problem 5. If $\mathcal{F}_i, i \in I$ for some index set I, are σ-algebras on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, show that $\mathcal{G} := \bigcap_{i \in I} \mathcal{F}_i$ is a σ-algebra.

It suffices to verify that the intersection of sets in the two original σ-algebras are in the new σ-algebra.

Problem 6 (optional)**.** Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where

$$\Omega = [0,1], \quad \mathcal{F} = \mathcal{B}_{[0,1]}, \quad \mathbb{P} = \text{Lebesgue measure},$$

where $\mathcal{B}_{[0,1]}$ denote the Borel σ-algebra on $[0,1]$. Consider the following two random variables defined on this probability space.

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in [0, 0.5) \\ 0, & \text{if } \omega \in [0.5, 1] \end{cases} \qquad Y(\omega) = \begin{cases} 1, & \text{if } \omega \in [0, 0.25) \cup [0.5, 0.75) \\ 0, & \text{if } \omega \in [0.25, 0.5) \cup [0.75, 1] \end{cases}$$

a) Check that X and Y have the same distribution.

b) Compute $\mathbb{P}[X = Y]$.

c) Write down the sets that are contained in $\sigma(X)$ and $\sigma(Y)$, respectively.

d) Show that X and Y are independent.

e) Explain how to define a sequence of random variables $X_1, X_2, \ldots$, so that (i) they all have the same distribution as X and (ii) they are mutually independent.

Problem 7 (optional, hard, requires book)**.** Consider a random variable X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}_1 \subset \mathcal{F}$ and $\mathcal{G}_2 \subset \mathcal{F}$ be two sub σ-algebras so that $\mathcal{G}_1$ is independent from $\sigma(\mathcal{G}_2, \sigma(X))$. Show that (this corresponds to property 5, equation (2.22) on pp. 45 of [Klebaner])

$$\mathbb{E}[X | \sigma(\mathcal{G}_1, \mathcal{G}_2)] = \mathbb{E}[X | \mathcal{G}_2], \quad \text{a.s.}$$

Remark. A definition for the *independence* of two σ-algebras (i.e., σ-fields) is given on page 39 of [Klebaner]. A random variable X is independent of a σ-algebra $\mathcal{G}$ if its generated σ-algebra $\sigma(X)$ is independent of $\mathcal{G}$.