MATH 545, Stochastic Calculus
Problem set 0

January 8, 2020

Except for problem 0, all these problems are optional and meant as a quick review of basic concepts in probability. If you want them corected they are due on THU Jan 15. You can give them to me in class, drop them in my box. In all of the problems $\mathbb{E}$ denotes the expected value with respect to the specified probability measure $\mathbb{P}$.

Problem 0. Read [Klebaner], Chapter 2 (by JAN 15) (terminology: σ-field = σ-algebra)

Problem 1 (optional)**.** I have \$50 and I'm gambling on a series of coin flips. For each head I win \$2 and for each tail I lose \$1. What's the probability that I will run out of money?

Problem 2 (optional)**.** Let $R(n)$ be a random draw of integers between 0 and $n-1$ (inclusive). I repeatedly apply R, starting at 10^{100}. What's the expected number of repeated applications until I get zero?

Problem 3 (optional)**.** Consider a Poisson process with parameter λ. What is the conditional probability that $N(1) = n$ given that $N(3) = n$? (Here, $N(t)$ is the number of calls which arrive between time 0 and time t.) Do you understand why this probability does not depend on λ?

Problem 4 (optional)**.** Let (X, Y) have joint density $\varrho(x, y) = e^{-y}$, for $0 < x < y$, and $\varrho(x, y) = 0$ elsewhere.

a) Are X and Y independent ?

b) Compute the marginal density of Y.

c) Show that $\varrho_X(x|Y = y) = 1/y$, for $0 < x < y$.

d) Compute $E(X|Y = y)$

e) Use the previous result to find $E(X)$.

Problem 5 (optional)**.** Let X, Y, and Z be independent uniformly distributed on $(0, 1)$.

a) Find the joint density of XY and Z^2.

b) Show that $\mathbb{P}\left[XY < Z^2\right] = \frac{5}{9}$.

Problem 6 (optional)**.** Consider a random walk S_i making $2n$ steps, and let T be the first return to its starting point, that is $T = \; min\{1 \leq k \leq 2n : S_k = 0\}$, and $T = 0$ if the walk does not return to zero in the first $2n$ steps. Show that for all $1 \leq k < n$ we have,

a) $\mathbb{P}[T = 2k] = \frac{1}{2k-1}\binom{2k}{k}2^{-2k}$

b) $\mathbf{E}\left[|S_{2k}| \,\middle|\, |S_{2k-1}| = r\right] = r$

c) $\mathbf{E}\left(|S_{2k+1}| \,\middle|\, |S_{2k}| = r\right) = 1$ if $r = 0,$ and $\mathbf{E}\left(|S_{2k+1}| \,\middle|\, |S_{2k}| = r\right) = r$ otherwise.

Problem 7 (optional)**.** A population contains X_n individuals at time $n = 0, 1, 2, \ldots$. Suppose that X_0 is distributed as Poisson(μ), whose probability distribution is given by

$$\mathbb{P}[X_0 = n] = e^{-\mu}\frac{\mu^n}{n!}\,.$$

Between time n and $n+1$ each of the X_n individuals dies with probability p independent of the others. The population at time $n+1$ is comprised of the survivors together with a random number of new immigrants who arrive independently in numbers distributed according to Poisson(μ).

a) What is the distribution of X_n ?

b) What happens to this distribution as $n \to \infty$? Your answer should depended on p and μ.

c) In particular, what is $\mathbb{E}[X_n]$ as $n \to \infty$?