Chapter 6

Problem 2: (a) If \( x = y \), setting \( x' = 0 \) yields \( x(1 + x^4) = \mu \) (which has a unique solution). The trace of the Jacobian associated with this system is \(-2\). Argue that, at the equilibrium for which \( x = y \), the determinant of the Jacobian has a sign change at some positive \( \mu = \mu^* \). In fact, \( \mu^* = \frac{4}{3\sqrt[4]{5}} \). If \( 0 < \mu < \mu^* \) the equilibrium is a sink and if \( \mu > \mu^* \) it’s a saddle.

(b) Sum the ODEs to see that \( y = \rho \) at equilibrium; the equilibrium value of \( x \) is \( \frac{\rho}{\sigma + \rho^2} \). Compute the Jacobian associated with the system. Assuming \( \sigma = \frac{1}{10} \), the determinant of the Jacobian is \( \rho^2 + \frac{1}{10} > 0 \). The trace of the Jacobian is given by

\[
\frac{2\rho^2}{\rho^2 + 1/10} - \rho^2 - \frac{11}{10},
\]

which is positive (implying instability) if \( \rho \in \left( \sqrt{\frac{2}{5} - \frac{1}{2\sqrt{5}}}, \sqrt{\frac{2}{5} + \frac{1}{2\sqrt{5}}} \right) \). For \( \rho \) in this range, the equilibrium is a source and a focus.

(c) and (d) See Sections 8.1.1 and 8.1.2 for solutions.

Problem 4: (a) Unstable. If \( x(0) > 0 \), no matter how small, the solution of the IVP grows until it asymptotes to \( x = \pi \).

(b) Asymptotically stable. For small \( x \), the term \(-x^3\) dominates \( \sin^2(x^2) \), which is \( O(x^4) \). Thus, the solution of any IVP for which \( |x(0)| \) is sufficiently small will decay to zero.

(c) Unstable. As may be seen in Figure 1,

\[
K = \{(x, y) : 0 \leq x \leq 1 \text{ and } 2x - 1 \leq y \leq 1\}
\]

is a trapping region, and \( x' < 0 \) in the interior of \( K \). Thus, if \( (x(0), y(0)) \in \text{Int}K \), then the solution of the IVP will move further from the equilibrium \((1, 1)\), eventually converging to the origin.

![Figure 1: See Problem 4(c).](image)

(d) Unstable. See Exercise 4c in Chapter 4. Remark: Since at the equilibrium \( DF_* = 0 \), the center manifold is the entire plane.

Problem 5: (a) If \( y \neq 0 \), the second equation in (6.78) implies that \( x = d_1/\rho_1 \), while if \( z \neq 0 \), the third equation in (6.78) implies that \( x = d_2/\rho_2 \).

(b) We will take our own advice and defer this problem until Chapter 8.
Problem 6: Write $A$ in Jordan normal form and use (2.35), the formula for the exponential of a Jordan block.

Problem 8: The first component of the solution of (6.59) is $x(t) = b_1 e^{-t}$. Thus, the second ODE has the particular solution $-(b_1^2 / 3) e^{-2t}$ and the general solution

$$y(t) = C e^t - b_1^2 / 3 e^{-2t}.$$ 

This solution converges to zero if and only if $C = 0$, and for the given initial conditions this happens iff $b_2 = -b_1^2 / 3$.

Problem 9: By symmetry, it suffices to consider initial conditions such that $b_2$ lies above $M_{g}$ (glob), i.e., such that $b_2 > -b_1 \sqrt{1 + b_1^2 / 2}$. As illustrated in Figure 6.9(a–c), any such trajectory is contained in a curve

$$\frac{x^2}{2} + \frac{x^4}{4} = \frac{y^2}{2} - C \tag{1}$$

where $C$ is the energy of the initial conditions. The particle moves with a nonzero speed that may be bounded from below. Therefore, for any such initial conditions both $x(t)$ and $y(t)$ tend to positive infinity as time evolves.

First let’s make rough estimates to fix the ideas of the proof. When $x$ and $y$ are large we may estimate the dominant behavior of these variables from (1), writing $x^4 / 4 \sim y^2 / 2$. Substituting this relation into the second equation in (6.61) we derive $dy/dt \sim ay^{3/2}$ where $a = 2^{3/4}$. Solutions of $dy/dt = ay^{3/2}$ of course blow up in finite time.

Our task is to make an actual argument from these ideas. For any initial conditions that lie above $M_{g}$ (glob), both $x$ and $y$ tend to infinity. Thus, there is a time such after which $x(t) > \sqrt{2}$ and $y(t) > 2\sqrt{|C|}$ where $C$ is the constant in (1); hence $x^4(t) / 4 > x^2(t) / 2$ and $y^2(t) / 4 - C > 0$. Using these inequalities, we derive that

$$\frac{x^4}{2} = \frac{x^4}{4} + \frac{x^4}{4} > \frac{x^2}{2} + \frac{x^4}{4} = \frac{y^2}{2} - C = \frac{y^2}{4} + \frac{y^2}{4} - C > \frac{y^2}{4}.$$

Then, substituting into the second equation in (6.61) we obtain

$$\frac{dy}{dt} > x^3 > ay^{3/2}$$

where $a = 2^{-3/4}$. Blowup in finite time follows by comparison.

**Remark:** Incidentally, solutions run backwards also blow in finite time if $b$ does not belong to $M_{g}$ (glob).

Problem 10: (a) Use (6.75) as a Hamiltonian.

(b) By the chain rule

$$\frac{d}{dt} H(x, y) = \sum_{j=1}^{d} \frac{\partial H}{\partial x_j} \frac{dx_j}{dt} + \sum_{j=1}^{d} \frac{\partial H}{\partial y_j} \frac{dy_j}{dt}$$

which, from (6.79), is zero.

(c) No need to work; just invoke Theorem 6.5.1.

(d) The basic content of this part of the problem, the remark at the end of Part (d), has nothing to do with Hamiltonian systems. Let us prove a more general result that includes Hamiltonian systems as a special case. Given two related systems of ODEs,

$$(a) \frac{dX}{ds} = F(X) \quad \text{and} \quad (b) \frac{dx}{dt} = \phi(x) F(x) \tag{2}$$

where $\phi(x) > 0$ is a smooth function that is bounded away from zero and infinity, we will show that these systems have the same orbits, even though trajectories have different parametrizations. (The use of different letters in (2a) and (2b) has no mathematical significance, but psychologically it may help reduce confusion in applying the chain rule.)
Consider an arbitrary solution of (2a), say $X_0(s)$. We introduce reparametrized time by writing $t = T(s)$ where
\[
T(s) = \int_0^s \frac{1}{\phi(X_0(s'))} \, ds'.
\]
Since
\[
\frac{dt}{ds}(s) = \frac{1}{\phi(X_0(s))}
\]
is nonzero, we may invert the definition, say $s = \tau(t)$ where $\tau(t) = T^{-1}(t)$, and we have
\[
\left. \frac{ds}{dt} \right|_{t=T(s)} = \left( \frac{dt}{ds}(s) \right)^{-1} = \left[ \frac{1}{\phi(X_0(s))} \right]^{-1} = \phi(X_0(s)). \tag{3}
\]
Now let
\[
x(t) = X_0(\tau(t)). \tag{4}
\]
By the chain rule
\[
\frac{dx}{dt}(t) = \frac{dX_0}{ds}(\tau(t)) \frac{d\tau}{dt}(t).
\]
For the first factor on the RHS we have from (2a) that
\[
\frac{dX_0}{ds}(\tau(t)) = F(X_0(\tau(t))) = F(x(t)).
\]
For the second factor $d\tau/dt$ or $ds/dt$, we invoke (3) with $s = \tau(t)$ to conclude
\[
\frac{d\tau}{dt}(t) = \phi(X_0(\tau(t))) = \phi(x(t)).
\]
Thus, $x(t)$ satisfies (2b). Moreover, as $t$ varies, $x(t)$ traces out the orbit of $X_0(s)$. Since these considerations apply to any orbits, equations (2a) and (2b) have the same orbits.

(e,f) In these equations, the multiplicative function $\phi(x, y)$ in Part (d) does not satisfy the constraints that were imposed on it in proving that (2a) and (2b) have the same orbits. However, along any compact subset of an orbit $\phi(x, y)$ is bounded away from zero and infinity, and this is sufficient.

**Problem 11:** Remark: This problem addresses a specific case of the general result in Problem 10(c).

**Problem 12:** (a) The set (6.82) is the region inside the bubble on the right in Figure 6.10(a). Show that the argument proving Theorem 6.5.3 applies for any initial conditions inside this region.

(b) If you are unsatisfied with the geometric ideas given in the hint, here is an analytical approach. By symmetry, it suffices to consider points on the right half of $\mathcal{M}_u$. The linearization of (6.1) at the origin has eigenvalues
\[
\lambda_\pm = \frac{-\beta \pm \sqrt{\beta^2 + 4}}{2}.
\]
The unstable eigenvector associated with $\lambda_+$ is $(1, c)$ where
\[
c = \frac{\sqrt{\beta^2 + 4} - \beta}{2}.
\]
Thus by Theorem 6.6.1, near the origin the equation for $\mathcal{M}_u$ has the expansion
\[
y = cx + \mathcal{O}(x^2).
\]
At such points the Hamiltonian equals
\[
H(x, y) = (c^2 - 1)x^2/2 + \mathcal{O}(x^3) < 0
\]
since $0 < c < 1$ (Show this!). Thus, $\mathcal{M}_u$ is contained in the set (6.82). Apply Part (a).

(c) The marble rolls back and forth across the origin many times till it comes to rest on top of the hill described by the potential $V(x) = -x^2/2 + x^4/4$.

**Problem 13:** (a) Show that the argument used to prove Theorem 6.5.2 applies to any initial conditions in $\mathcal{U}$.

(b) Let $\mathbf{F}(x,y,z)$ denote the right-hand side of the Lorenz equations (6.76). After some algebra (including completing the square),

$$
\langle \nabla L(x, y, z), \mathbf{F}(x, y, z) \rangle = 2 \left[ -\left( x - \frac{\rho + 1}{2} y \right)^2 - \left( 1 - \left( \frac{\rho + 1}{2} \right)^2 \right) y^2 - \beta z^2 \right],
$$

which is negative away from the origin if $\rho < 1$.

**Problem 15:** The $x$-nullcline $y = x$ (shown in brown) crosses the $y$-nullcline (shown in cyan) at $(0,0)$ (a saddle equilibrium) and $(1,1)$ (a sink). The stable and unstable manifolds of $(0,0)$ are shown in blue and red, respectively.

**Problem 18:** See the clarifications to this problem in the errata.

**Problem 20:** As in Section 6.3.2, $\mathbf{D}\mathbf{F}_*$ may be transformed to block diagonal form (6.30), where now

$$
B = \begin{bmatrix}
D_1 & 0 \\
0 & D_2
\end{bmatrix}.
$$

All eigenvalues of $\mathbf{D}\mathbf{F}_*$ have negative real parts unless $\det(A - 2B) < 0$. For positive diffusion coefficients, this determinant is negative when

$$D_1 < 1 \quad \text{and} \quad D_2 > \frac{\rho(2x_+\sigma - 1)}{1 - D_1}.$$ 

The second inequality may be manipulated to a form similar to the displayed formula above (6.31).

**Problem 21:** See the [minor] correction to the suggestion for this problem in the errata.