Optimal Overbooking

Team 180

February 11, 2002

1 Introduction

In our paper, we construct several models to examine the effect of different overbooking policies on airline revenue and costs in light of the current state of the industry. During the past several months, airlines have experienced a decrease in their flight volume, an increase in security on their premises, heightened passenger fear, and billions of dollars worth of financial loss. We refer to these four influences as the "complicating factors." After presenting some basic terms and our most important assumptions, we begin our analysis with a discussion of the qualitative effects the aforementioned factors are likely to have on airline revenue and auxillary factors that affect it.

The costs that airlines have to pay to passengers who are bumped off of flights for which they have already purchased tickets affect revenue substantially. At the same time, airlines stand to increase their revenue by filling their flights closer to capacity. When passengers get bumped as a result of overbooking, they can either choose or refuse to negotiate with the airlines for compensation. These choices pose different two diffent types of costs to the airline.

In the case where ticketholders are bumped involuntarily, the Department of Transportation specifies a set of guidelines for mandatory compensation. The compensation formulas given, however, depend nonlinearly on both the ticket price and the time a bumped passenger must wait to catch the next flight to his or her destination. Using a plausible value for average ticket price based on our research, we present a model for the waiting time distribution that allows us to estimate the average cost per involuntarily bumped passenger.

In the case where ticketholders are willing to be bumped voluntarily, the interaction between the airline and ticketholders takes the form of a least bid auction in which winners receive compensation for forgoing their flights.

We discuss the precedent for this type of auction and introduce a highly similar continuous auction model. This continuous auction model allows us to calculate a novel formula for the expected amount of compensation required by a given group of voluntarily bumped passengers.

All bumped passengers are either bumped voluntarily or involuntarily, and our analysis of the expected costs involved in these two instances, along with research into the relative frequency of these alternatives, leads us to a good understanding of the costs associated with bumped passengers.

To more fully develop an understanding for the relationship between the complicating factors mentioned above and the optimal overbooking policy, we developed two very different but complimentary models: The One Plane Model and The Interactive Simulation Model.

The One Plane Model models expected revenue as a function of overbooking policy in the one plane case. Using this framework, we examined the relationship between the optimal (revenue maximizing) overbooking strategy and the arrival probablity of ticketholders. We then extended the one plane model to consider multiple fare classes. This analysis revealed, among other things, that the inclusion of multiple fare classes does not significantly alter optimal overbooking policy. Drawing on our preliminary analysis of the complicating factors, we assessed the likely changes that would occur in arrival probability of ticketholders and demonstrated the possible effects these would have on optimal booking.

The Interactive Simulation Model is our most comprehensive model, which takes into account, among other things, our estimates for average compensation costs. In particular, this model simulates the interaction between 10 major US airlines with a market base of 10,000 people, factoring in passenger arrival probability, flight frequency, compensation for voluntary bumping, compensation for involuntary bumping, and the behavior of rivals in terms of altered overbooking policy. We use this model to estimate optimal booking policy in a competitive environment, which has the significant advantage of factoring in passenger switching between airlines. We ran a large simulation of our Interactive Model with what we surmised, based on our previous analysis, were likely parameter values before and after September 11 (which exacerbated the problems of the airlines through all of the complicating factors). Both simulations contained many trials and gave robust results that corroborated with our conclusions based on the One Plane Model and the Involuntary and Voluntary Compensation Cost Models.

Overall, we conclude that airlines should keep their current levels of overbooking or decrease them.

2 Terms

- **Ticketholders**: people who have purchased a ticket and whose individual ticket revenue the airline has already received.
- **Contenders**: ticketholders who arrive at the terminal in time to board the flight for which they have purchased a ticket.
- **Boarded Passengers**: contenders who are able to board their flight successfully.
- **Bumped Passengers**: contenders who are not given seating on their flight.
- **Voluntarily Bumped Passengers**: bumped passengers who opt out of their seating in exchange for some sort of compensation (usually monetary) by the airline.
- **Involuntarily Bumped Passengers**: bumped passengers who are denied boarding against their will.
- **Compensation Costs**: the total value of money and other incentives given to bumped passengers by airlines.
- **Flight Capacity**: the total number of seats on a given flight.
- **Overbooking**: the practice of selling a number of tickets for a flight that is greater than the flight capacity.
- **Waiting Time**: the time a bumped passenger would have to wait to catch the next flight to his or her destination.
- **Load Factor**: the ratio of the number of seats filled to the capacity.

3 Assumptions and Hypotheses

- Flights are domestic, direct, and one-way.
- The waiting time between flights equals the amount of time until the scheduled departure time of the next available flight to a given destination.
- The ticket price is $140 [ATA website, www.air-transport.org], except when we consider multiple fares in Section 6 (???).

- The ticket price is independent of the time at which the ticket is bought.
- Pre-September 11, the average probability of a ticketholder actually checking in for his flight (and thus becoming a contender) was 85% [Smith et. al, 9].
- The pre-September 11 average load factor was 72% [Bureau of Transportation Statistics wesite, www.bts.gov].

4 Preliminary Discussion of the Complicating Factors

Each of our models attempts to take into account the current situation facing airlines. We will refer collectively to the four issues mentioned in the problem statement as the "complicating factors." Individually, they will be referred to as

- **The Traffic Factor**
  There are, on average, fewer flights by airlines between any given locations throughout the day and night.
- **The Security Factor**
  Security in and around airports has been heightened in the wake of September 11th.
- **The Fear Factor**
  Passengers are more wary of the dangers of air travel, such as possible terrorist attacks, plane crashes, and security breaches at airports.
- **The Financial Loss Factor**
  Airlines have lost billions of dollars in revenue during the past several months due to decreased demand for air travel, increased security costs, and increased industry risks.

Before presenting our models and the modifications we have made to handle effects due to these factors, we begin with a preliminary analysis of the implications of each.

The Traffic Factor

Because there are fewer flights between any two locations on average, it is likely that–all other things being equal– the demand for any given flight will increase. In keeping with this trend, flights are likely to be fuller than before, and the average waiting time between flights to a given destination is likely to increase. Because the average waiting time between flights is likely to increase, it is reasonble to expect that when bumped, passengers will demand a higher level of compensation for their troubles.

In the case of passengers who choose to accept some sort of compensation for being bumped, the average requested price for acquiescing is likely to be higher. For involuntarily bumped passengers, the Department of Transportation specifies guidelines for compensation that depend on ticket price and the amount of time the passenger must wait for the next available flight. These fees are also likely to be higher, for two reasons. First, assuming the supply of flights falls more than the decrease in demand for flights to all destinations, the price of tickets is likely to rise. Second, the increase in average waiting time in likely to make involuntary compensation more costly for the airlines. We analyze the cases of voluntarily and involuntarily bumped passengers in detail using auction models, the structure of DOT regulations, and a model of the waiting time distribution.

In light of the traffic factor effects just mentioned, it is possible that of the contenders for an overbooked flight, fewer will voluntarily give up their seats. If true, this would imply an increase in the cost of involuntary compensation.

The Security Factor

The increase in the level of security in and around airports will likely lead to an increase in the number of ticketholders who arrive at the airport but, due to security delays, are unable to arrive at their departure gates in time. This effect will lead to a decrease in the probability p that an individual ticketholder will show up for his or her flight. We address the implications of this fact in our discussion of the One Plane Model and the Interactive Simulation Model.

Additionally, successful implementation of security measures may lead to an improvement in the public perception of the airline industry and an increase in the demand for air travel.

The Fear Factor

The fear factor works, in part, to offset the effects of the security factor. On one hand, increased fear of flying leads to a decrease in the demand for air travel and a decrease in the market size. If fewer people decide to fly, then security delays may not be as serious. On the other hand, if a higher percentage of ticketholders are flying for reasons of necessity, then the probability that a ticketholder will become a contender may increase because of a decrease in cancellations and no-shows.

In terms of its effect on the behavior of bumped passengers, the fear factor implies that, because a higher portion of passengers are flying out of necessity, fewer will agree to be bumped voluntarily at any price. Thus, the percentage of involuntarily bumped passengers may increase. Along with this, the average level of compensation required by voluntarily bumped passengers will probably increase.

The Financial Loss Factor

Because companies may be seeking to increase short-term profits in the face of recent losses, it is possible that some airlines may decide to implement more aggressive overbooking policies. Should such a situation occur, it might induce an overbooking war between airlines if increased overbooking leads to higher revenue from planes being filled closer to capacity. Other have also suggested this idea [Suzuki, 148]. Airlines who had not implemented these policies would have a short term incentive to follow suit to match their competitors. The likely increase in the number of bumped passengers, of course, would lead to a rise in compensation costs that would offset the increased revenue. If airlines uniformly decide to implement aggressive overbooking policies, the net result would probably hurt most or all of the airlines through higher compensation costs.

If airlines are less myopic, they will also take into account the effect that their public image has on demand for air travel. In particular, decreasing the number of bumped passengers would help improve their image and might spur demand, which would bolster their future revenue stream. In practice, it is important for individual airlines to assess the relative importance of these effects when setting their overbooking policy.

5 One Plane Model

Introduction and Motivation

We first consider the optimal overbooking strategy for a single flight, independent of all other flights. This is a substantial simplification of the main overbooking problem. The primary reason we build the One Plane model is as a first step towards the much larger Interactive Simulation Model we present later. That being said, the One Plane Model is itself useful for several reasons. First, it serves to develop intuition about the general overbooking problem. Second, it considers a nontrivial case which is still very tractable, and thus allows for a great deal of analysis. Finally, we will see later that its results are a good approximation to the results of the full-fledged Interaction Simulation Model.

Development

We model the expected revenue as a function of the overbooking strategy for the single plane, and find the revenue-maximizing strategy.

We assume that the plane has capacity of C identical seats. This assumption is relaxed later, when we consider a multi-fare model. We assume also that a ticket costs $T = \$140$ independent of the time at which it is bought. Finally, we assume that the airline's overbooking strategy is to sell up to B tickets, if possible ($B > C$).

We analyze this strategy in the case where the flight sells out completely (i.e. all B tickets are sold). Analyzing this case is one of the most direct ways to gauge the effectiveness of the company's overbooking strategy.

We model the number of contenders for the flight with a binomial distribution, where a ticketholder becomes a contender with probablity p. According to Smith et al., the average p value for flights from the ten leading US carriers is $p = 0.85$. Note however, that the p value for a particular flight depends on a host of factors - for example, flight time, length, destination, and whether it is a holiday season. Because of the potential p variation from flight to flight, we carry out our analysis for a range of possible p values. However, a real airline company has, or could easily obtain, an empirical value of p for any particular flight (e.g. the noon New York to Boston flight in August). An airline can use its empirically determined p in the models we are about to develop.

With our binomial model, the probablity that there are exactly i contenders among the B ticket-holders is

$$\binom{B}{i}p^i(1-p)^{B-i}.$$

Next, we model compensation costs. We assume that each bumped passenger is paid a constant compensation cost of $(k+1)T = 140(k+1)$, for some positive constant k. Translated into everyday terms, this means that a bumped passenger receives compensation equal to his ticket price and then some additional compensation $kT > 0$. The assumption that compensation cost is constant for each bumped passenger is relaxed later, when we consider involuntary versus voluntary bumping.

We define the compensation cost function $F(i, C)$ to be the total compensation the airline must pay if there are exactly i contenders for a flight with seating capacity C:

$$F(i,C) = \begin{cases} 0 & i \leq C \\ (k+1)T(i-C) & i > C \end{cases}$$

With the results we have so far, we now calculate expected revenue, R, as a function of the overbooking strategy B:

$$\begin{aligned} R(B) &= \sum_{i=1}^{B} \binom{B}{i} p^i (1-p)^{B-i} (BT - F(i,C)) & (1) \\ &= 140B - 140(k+1) \sum_{i=C+1}^{B} \binom{B}{i} p^i (1-p)^{B-i} (i-C) & (2) \end{aligned}$$

Thus, for given C, p, and k, it is possible to determine the overbooking strategy B_{opt} which maximizes R(B). We do this with a computer program for various C, p, and k. On the other hand, it is possible to produce a close analytic approximation to this data. Our program now is first to derive this approximation, and then to interpret the computer-obtained data.

We begin by noting that the revenue for a bumped passenger, $T - (k+1)T = -kT$, has magnitude k times that for a boarded passenger, T. Thus, it was hypothesized that the optimal overbooking strategy should be chosen such that the distribution of contenders is in some sense "balanced," with $1/(k+1)$ of its area corresponding to bumped passengers and the remaining $k/(k+1)$ corresponding to boarded passengers.

We approximate the binomial distribution of contenders with a standard normal distribution so that we may make use of the function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2} dt$$

to obtain

$$\frac{C - Bp}{\sqrt{Bp(1-p)}} = \Phi^{-1}(\frac{k}{k+1}),$$

which we may write since Φ is one-to-one. Clearing denominators and solving the resulting quadratic in $\sqrt{B}$ gives

$$B'_{opt} = \left(\frac{-\Phi^{-1}(\frac{k}{k+1})\sqrt{p(1-p)} + \sqrt{\Phi^{-1}(\frac{k}{k+1})^2 p(1-p) + 4pC}}{2p} \right)^2 \qquad (3)$$

as an our analytic approximation to B_{opt}. Note that for $k = 1$, this gives $B'_{opt} = C/p$.

Results and Interpretation

Utilizing a simple computer program, we solved Equation (2) for the optimal overbooking strategy B for certain k, C, and p. See the appendix for full results and code. In Table 3 we present the results for plane capacity $C = 150$, as compared with the formula (3).

p	k	B_{opt}	B'_{opt}
0.80	1	189	188
0.85	1	177	176
0.90	1	167	167
0.80	2	186	185
0.85	2	175	174
0.90	2	165	165
0.80	3	184	183
0.85	3	173	173
0.90	3	164	164

Table 1: Optimal overbooking strategy versus the mathematical approximation B'_{opt} for selected arrival probabilities and compensation constants

Note that in real life k will almost never exceed 3. That is, bumped passengers will almost never be compensated by $k+1 = 4$ times their original ticket price. A more likely compensation cost (c.f. our dicussion in Section 7) corresponds to $k = 1$ or 2. Note also that, as mentioned earlier, the average p for all US carriers is about $p = 0.85$. These realistic values of p and k are considered in Table 1, and for them, formula (3) yields suggested overbooking rates very close to, though somewhat more conservative than the true, computer-determined optimal values. Thus this formula may be considered as a reasonable approximation to the true B_{opt}.

We now analyze the computer-generated data for the optimal overbooking strategy in further detail. Note that for a given flight, C and T are obviously known, and that, as explained earlier, the airline should be able to obtain very good approximations to p and k empirically. Thus, an airline can take our computer program, insert its data for C, T, p, and k, and obtain the optimal overbooking strategy B_{opt}. Figure 1 plots expected revenue $R(B)$ versus overbooking strategy B according to Equation (2) for $C = 150$, $k = 1$, $p = 0.85$, and $T = 140$.

At the optimal overbooking strategy of $B = 177$, the airline can expect to make revenue $R(177) = \$24,200$, which is more than 15% in excess of the expected revenue $R(150) = \$21,000$ resulting from a policy of no over-booking whatsoever. This shows the obvious advantages to airlines of overbooking.

More importantly, however, operating at a less-than-optimal overbooking strategy can have serious consequences for an airline's revenue. For example, American Airlines has an annual revenue of approximately $20 billion [AMR Corp. website, www.amrcorp.com]. A quick calculation using our model shows that an overbooking policy B outside the range of $[173, 183]$ implies an expected loss of more than one billion dollars over a 5-year period from the expected revenue at $B_{opt} = 177$. This shows the financial impact of choosing optimal or near-optimal overbooking strategies as opposed to less optimal ones.

Limitations of the Basic One Plane Model

The single-plane model developed so far is admittedly somewhat simplistic. Its most obvious limitation is failing to account for bumped passengers' general dissatisfaction and propensity to switch airlines. Also, the model assumes a simple constant-cost compensation function for bumped passengers. Along similar lines, the model ignores the distinction between voluntary and involuntary bumping. Additionally, the basic model assumes that all tickets are identical - that is, everyone flies coach. Also, the model assumes that all B tickets the airline is willing to sell are actually sold, which is not the

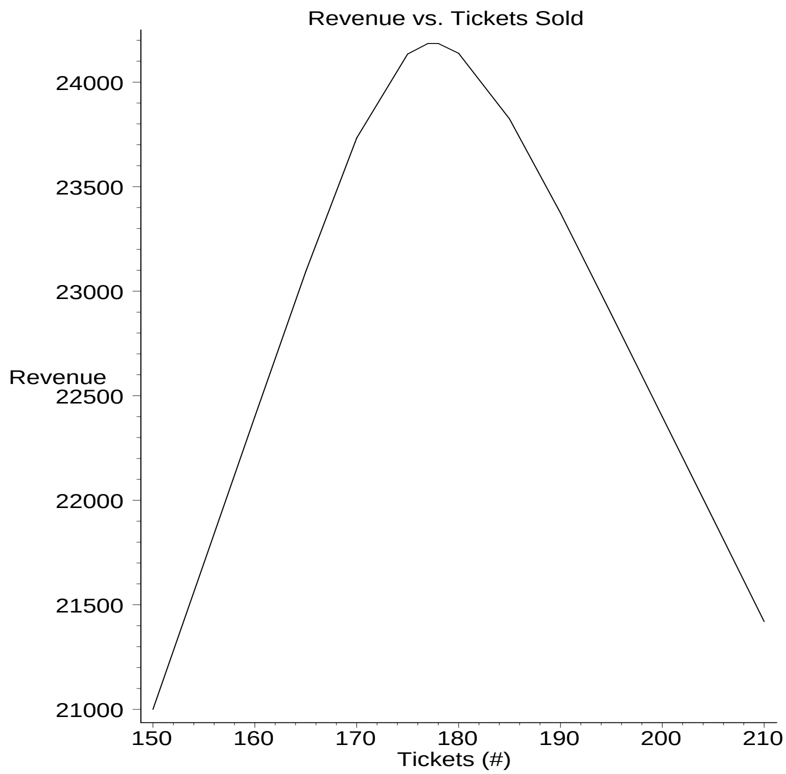


Figure 1: Revenue R versus overbooking strategy B for $C = 150$, $k = 1$, $p = 0.85$, and $T = \$140$

case for every flight.

Even so, the model developed thus far does successfully analyze some of the most important variables in the overbooking problem - revenue as a function of overbooking strategy when the plane capacity, the probablity that ticket-holders become contenders, and the compensation cost function vary. Later, we develop a much more complete model which fully incorporates many of the variables the basic model ignores.

The Complicating Factors

First though, we utilize the basic model to make preliminary predictions for the optimal overbooking strategy in light of market changes due to the complicating factors post-September 11. Later we will use the Interactive Simulation Model to make much more extensive predictions.

Of the four complicating factors, only two are directly relevant to this model: the security factor and the fear factor. According to our analysis in the preliminary discussion section, the primary effect of the security factor is to decrease the probability p of a ticketholder reaching the gate on time and becoming a contender. On the other hand, the primary effect of the fear factor is that a greater proportion of those who fly do so out of necessity. Since such passengers are more likely to arrive for their flights than more casual flyers, the fear factor tends to increase p.

Figure 2 plots the optimal overbooking strategy B_{opt} versus p (for fixed $k = 1$ and $C = 150$) as produced by our model.

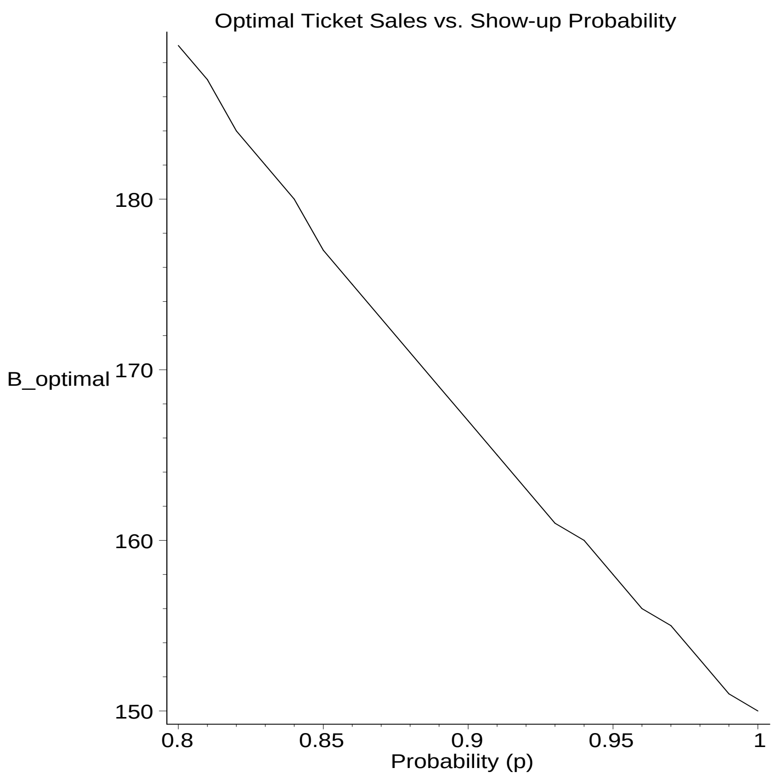


Figure 2: Optimal overbooking strategy versus arrival probability p

It is difficult for us to assess the precise change in p resulting from the security and fear factors. However, airlines can determine this empirically with relatively little difficulty by gathering statistics on their own flights. Then they can use our graph or computer program to determine their new optimal overbooking strategy according to the One Plane Model.

Conclusion

The One Plane Model provides a definitive answer for the revenue-maximizing overbooking strategy in certain scenarios. In particular, it produces the value B_{opt} for the maximum number of tickets the airline should sell for a particular flight. The model can obtain this answer under varying conditions of plane size C, arrival probability p, and compensation cost constant k. For example, under similar-to-life conditions of $p = 0.85$ and $k = 1$, the Model suggests selling up to 177 tickets for a 150-seat plane in order to maximize revenue. Additionally, we produced a formula which furnishes a good approximation to the true computer-generated B_{opt}.

The Model is useful because it allows for a great deal of reasonably accurate analysis in an important subcase of the general overbooking problem. However, the Model does not address situations involving multiple flights or multiple airlines. The Interactive Simulation Model which we present later does consider these factors.

6 One Plane Model: Multi-Fare Extension

Introduction and Motivation

Most airlines sell tickets in different fare classes, which are most commonly first class and coach. In this section, we extend the basic One Plane Model, which considered a flight with only one fare class, to account for mutiple fare classes. One important goal of this section is to assess the importance of including multiple fare classes in the Interactive Simulation Model which we build later.

Development

For simplicity, we consider a two-fare system, with first class and coach tickets for a single plane. The plane has C_1 first class seats and C_2 coach seats. We assume that a first class ticket costs $T_1 = \$280$ and that a coach ticket costs $T_2 = \$140$. We consider an overbooking strategy of selling up to B_1 first class tickets and up to B_2 coach tickets, where the two types of sales are made independently of one another.

Similar to before, we assume that a first class ticketholder becomes a first class contender with probability p_1 and that a coach ticketholder becomes a coach contender with probability p_2. We use two independent binomial distributions as our model. First class ticketholders are more likely to become contenders than coach passengers since they have made a larger monetary investment in their tickets. That is, $p_1 > p_2$.

Thus, the probability that there are exactly i first class contenders is

$$\binom{B_1}{i} p_1^i (1-p_1)^{B_1-i}$$

and the probablity that there are exactly j coach contenders is

$$\binom{B_2}{j} p_2^j (1-p_2)^{B_2-j}.$$

We model compensation costs as constant per bumped passenger but dependent on fare class, with $(k_1+1)T_1$ as compensation for a bumped first class passenger and $(k_2+1)T_2$ for a bumped coach passenger, for some positive k_1 and k_2. Next, we define the compensation cost function $F(i, j, C_1, C_2)$ to be the total compensation cost the airline pays if there are i first class and j coach contenders for a plane with C_1 first class and C_2 coach seats:

$$F(i,j,C_1,C_2) = \begin{cases} 0 & i \le C_1,\ j \le C_2 \\ T_1(k_1+1)(i-C_1) & i > C_1,\ j \le C_2 \\ \max\{T_2(k_2+1)((j-C_2)-(i-C_1)), 0\} & i \le C_1,\ j > C_2 \\ T_1(k_1+1)(i-C_1) + T_2(k_2+1)(j-C_2) & i > C_1,\ j > C_2 \end{cases}$$

The justification for the third case is that an excess of coach contenders is allowed to spill over into any available first class seats. Note on the other hand that excess first class contenders cannot be seated in any available coach seats; this is reflected in the second case.

We can now model expected revenue, R, as a function of the overbooking strategy (B_1, B_2).

$$\begin{aligned} R(B_1, B_2) \quad = \quad & \sum_{i=1}^{B_1} \sum_{j=1}^{B_2} \binom{B_1}{i} \binom{B_2}{j} p_1^i (1-p_1)^{B_1-i} p_2^j (1-p_2)^{B_2-j} \cdot \\ & (B_1 T_1 + B_2 T_2 - F(i,j,C_1,C_2)) \end{aligned}$$

Results and Interpretation

For fixed C_i, T_i, p_i, and k_i $(i = 1, 2)$, we can find $(B_{1,opt}, B_{2,opt})$ for which $R(B_1, B_2)$ is maximal by adapting the computer program used to solve the one-fare case. Note that for a given flight, the C_i and T_i are obviously known, and moreover that the airline can obtain values of the p_i and k_i empirically (c.f. the discussion in section 5.3).

For example, for a plane with $C_1 = 20$ first class seats, $C_2 = 130$ coach seats, ticket costs of $T_1 = \$280$ and $T_2 = \$140$, and compensation constants $k_1 = k_2 = 1$, we obtain the optimal overbooking strategies listed in Table 2.

p_1	p_2	$B_{1,opt}$	$B_{2,opt}$
0.85	0.80	23	165
0.90	0.80	22	165
0.95	0.80	20	166
0.85	0.85	23	155
0.90	0.85	22	155
0.95	0.85	20	155
0.90	0.90	22	146
0.95	0.90	21	145

Table 2: Two-fare optimal overbooking strategies for selected arrival probabilities

The optimal strategy involves relatively little overbooking of first class passengers. This is a reasonable result since there is a much higher compensation cost for bumping first class passengers versus bumping coach passengers.

How Important is Modeling Multiple Fare Classes?

We note that, for constant p_2, the total number of overbooked passengers $B_{1,opt} + B_{2,opt}$ in the optimal strategy is not perturbed much by the specific value of p_1. This fact is shown in the Table 3. It compares the total number of overbooked passengers $B_{1,opt} + B_{2,opt}$ in the optimal two-fare strategy for arrival probabilities (p_1, p_2) with the number of overbooked passengers B_{opt} in the optimal one-fare strategy for arrival probability $p = p_2$. That is, the arrival probability for the one-fare case is set equal to the *coach* arrival probability for the corresponding two-fare case.

Table 3 clearly shows that $B_{1,opt} + B_{2,opt}$ is equal to or very nearly equal to B_{opt} in all cases. The upshot is that the effect of multiple fare classes on the optimal overbooking strategy is not very significant. As a result, when we construct our more general model later, we do not take into account multiple fares.

p_1	p_2	$B_{1,opt} + B_{2,opt}$	B_{opt} for $p = p_2$
0.85	0.80	188	189
0.90	0.80	187	189
0.95	0.80	186	189
0.85	0.85	178	177
0.90	0.85	177	177
0.95	0.85	175	177
0.90	0.90	168	167
0.95	0.90	166	167

Table 3: Total number overbooked in optimal two-fare versus one-fare strategies

Conclusion

The Extended One Plane Model successfully extends the basic One Plane Model to account for multiple fare classes. The results of the Extended Model are very reasonable; it suggests relatively little overbooking of first-class passengers and much more extensive overbooking in coach. Finally, based on the results of the Extended One Plane Model, specifically Table 4, we decide not to model multiple fare classes later, in the Interactive Simulation Model.

7 Analysis of Compensation Costs

In this section, we analyze the cost to airlines of compensating bumped passengers. There are many schemes for dealing with passengers who are in danger of being bumped from a flight for which that have purchased a ticket. The key element that seperates different schemes is the degree of choice involved for the passenger. In the event that a passenger is completely unwilling to relinquish the right to his or her seat on an overbooked flight, the airline may have to forcefully suspend the use of the passenger's ticket. In such events, the passenger has little choice in the amount of compensation he or she is entitled to receive, but guidelines for handling this situation have been established by the Department of Transportation.

On the other hand, a passeger may agree to compromise with the airline in the hope of extracting a price for his or her troubles. This way of dealing with being bumped involves a great deal more choice than the former scenario, and often can lead to a favorable compromise. To accomplish this, airlines often hold auctions for contenders in which the lowest bids are first to be bought off of a flight.

In this section, we divide our analysis of bumping costs into two parts. First, we construct a model for involuntary bumping costs based on DOT regulations, which takes into account the waiting time distribution for flights. After this, we discuss auction methods for voluntary bumping and derive novel results for expected compensation cost for a continuous auction that matches actual ticket auctions fairly well.

Our purpose in analyzing both the voluntary and involuntary aspects of bumping–via regulation schemes and auctions–is to build useful and accurate models of the costs of bumping passengers. We use results from both compensation cost models in our Interactive Simulation Model to help produce informative data on the effect of the complicating factors on optimal bumping policy.

Involuntary Bumping: DOT Regulations

The Department of Transportation (DOT) requires each airline to give all passengers who are bumped involuntarily a written statement describing their rights and explaining how the airline decides who gets on an overbooked flight and who does not [DOT Website, www.dot.gov]. Those travellers who do not get to fly are usually entitled to an "on-the-spot" payment of denied boarding compensation. The amount depends on the price of their ticket and the length of the delay in the following manner:

- Passengers who are bumped involuntarily for whom the airline arranges substitute transportation that is scheduled to get to their final destination (including later connections) within one hour of their original scheduled arrival time receive no compensation.

- If the airline arranges substitute transportation that is scheduled to arrive at the destination between one and two hours after the original arrival time, the airline must pay bumped passengers an amount equal to their one-way fare, with a $200 maximum.

- If the substitute transportation is scheduled to get to the destination more than two hours later, or if the airline does not make any substitute travel arrangements for the bumped passenger, the airline must pay an amount equal to the smaller of 200% of the fare price and $400.

- Bumped passengers always get to keep their tickets and use them on another flight. If they choose to make their own arrangements, they are always entitled to an "involuntary refund" for their original ticket.

These conditions apply only to domestic flights, and do not apply to planes that hold 60 or fewer passengers. There are a few other minor exceptions, but they are not important for our purposes. By aggregating the above information, we find that the function representing the compensation cost for an involuntarily bumped passenger is

$$C(T,F) = \begin{cases} 0 & \text{if } 0 < T \leq 1 \\ \min(2F, F+200) & \text{if } 1 < T \leq 2 \\ \min(3F, F+400) & \text{if } 2 < T \end{cases}$$

where T is waiting time and F is the fare price. As stated before, we assume that all flights to a given location are direct and have the same flight durations. Thus, the waiting time between flights equals the difference in departure times, and the waiting time T is understood to be the time until the next flight to a given destination departs. In formulating the above equation, we assume that involuntary passengers always ask for a refund of their fare.

Involuntary Bumping: The Waiting Time Model and Expected Cost

In order to use the compensation cost function above to determine the average compensation (per involuntarily bumped passenger), we would need to know the joint distribution of fare prices and waiting times. Because this information would be extremely difficult to obtain with full accuracy for a particular airline during a given day or month, we opt instead to make a couple of practical compromises:

1. First, we begin by restricting our attention to determing the expected compensation cost for the average ticket price of $140. This figure was obtained from Air-Transport data for the year 2000 [www.air-transport.org].

2. Second, we specify a workable model for the distribution of waiting times that allows us to calculate this cost directly.

Justification

Our model for the distribution of waiting times will be the exponential distribution. There are three primary reasons to recommend the use of this model:

1. The exponential distribution often arises, in practice, as being the distribution of the amount of time until some specific event occurs [Ross, 217].

2. After looking up the flight schedules for several major airlines on the internet and observing the distribution of flight times between several large cities, the exponential distribution seemed to provide a tolerably good fit.

3. If the waiting times between events are distributed according to an exponential distribution with parameter λ, the number of events that occur within one time unit follows a Poisson process with parameter λ. Plane arrivals could be represented by such a process. The Poisson process has the property that the sum of two Poisson processes with parameters λ_1 and λ_2, respectively, is a Poisson process with parameter $\lambda_1 + \lambda_2$. This property coincides more closely with what we might expect from the distribution of plane arrivals than, say, a normal distribution, and further evidence is lent by the fact that the distributions of plane arrivals on different days tend to be highly similar.

The Exponential Model

To state the exponential model formally, let T be a random variable representing waiting time between flights such that

$$Prob(T \leq t) = 1 - e^{-\lambda t}.$$

By the properties of the exponential distribution, we know that $E(T) = \tau = 1/\lambda$, where τ is the mean waiting time for the next available flight. In reality, τ is a function of many factors, including the time of year and the attractiveness of the destination. In particular, the amount of security delays being experienced by the airport system (the security factor) and the fact that, at the present time, the volume of flights has on average decreased (the traffic factor) both work to increase τ above previous levels.

Expected Cost

With this framework in place, we can compute the expected cost of compensating an involuntarily bumped passenger.

Proposition 7.1. *Assuming that wait time T has an exponential distribution with parameter λ, and that compensation cost is a function of fare price and wait time given by $C(F,T)$, the expected compensation cost to an involuntarily bumped passenger who has purchased a ticket of price $F = P$ is*

$$\min(2P, P+200)[e^{-\lambda} - e^{-2\lambda}] + \min(3P, P+400)[e^{-2\lambda}].$$

Proof. An exponential random variable with parameter λ has a probability density function equal to $\lambda e^{-\lambda t}$. Following from the definition of conditional expectation and the definition of $C(F,t)$, we have

$$\begin{aligned} E[C(F,t)|F=P] &= \int_0^\infty \lambda e^{-\lambda t} C(P,t)dt \\ &= \int_0^1 \lambda e^{-\lambda t}(0)dt \\ &\quad + \int_1^2 \lambda e^{-\lambda t} \min(2P, P+200)dt \\ &\quad + \int_2^\infty \lambda e^{-\lambda t} \min(3P, P+400)dt. \end{aligned}$$

Evaluating the above integrals in t yields the desired result. □

Having arrived at this result, we have a basis to estimate the average compensation cost per involuntarily booked passenger.

Estimating τ

The only task remaining is to estimate the average waiting time τ. Unfortunately, we were not able to locate any waiting time statistics that would indicate a plausible value for τ. However, in the absence of such data, we conducted several searches of online booking sites for flights between major cities. By our informal calculations, a reasonable daytime value for τ is approximately 2.6 hours. This calculation omits the time between the last flight of the day and the first flight of the next day to a particular destination. If we include these night-next-day waiting times in our calculations, we obtain the value $\tau \approx 4.8$. In particular, this value for τ corresponds to the frequency of 5 flights per 24 hour period, which is fairly typical. Using the smaller, strictly daytime value $\tau = 2.6$ hours, we obtain an expected compensation cost of:

$$E[C(F,t)|F=140] = \$255$$

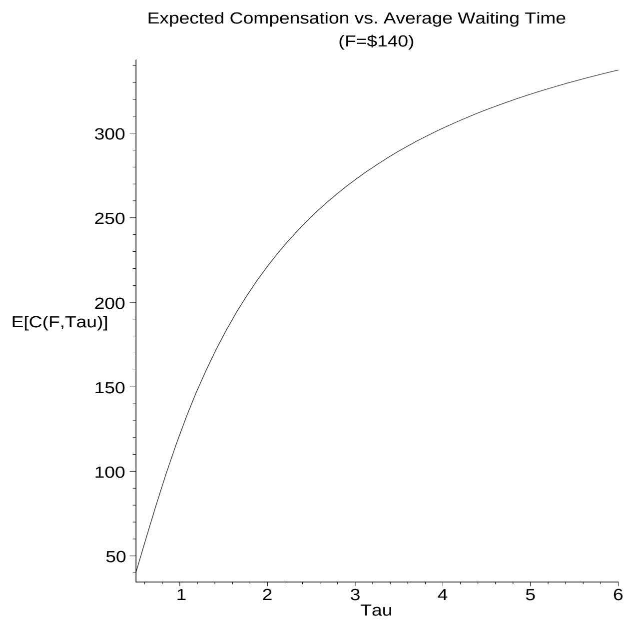


Figure 3: Expected Involuntary Compensation Cost vs. Average Waiting Time (F = $140)

This is the value for involuntary compensation cost that we use as an input into our Interactive Simulation Model. Figure **??** displays the expected compensation cost for several values of average waiting time. Our estimate is at odds with another source in the literature, which quoted an average compensation amount of $50 in 1997 dollars [Alstrup et al., 1986]. To illustrate the extent to which our estimates differ, assume that $50 is an accurate figure; holding the average ticket price constant at $140, our expected compensation cost model implies that the average waiting time would have to be approximately $\tau = 33$ minutes. This waiting time obviously does not reflect reality.

Evaluation and Commentary: Cost Implications

One possible conclusion is that our model does not provide the best fit for the waiting time distribution. This is entirely possible, although our informal examination of actual departure time data seemed to indicate that it was a decent fit, and our other two reasons for using this distribution carry some weight as well. Nevertheless, more work needs to be done to gain a full understanding of the waiting time distribution.

Another possible conclusion we might draw from this discrepancy is that involuntary bumping regulations were changed or not enforced properly 15 years ago when Alstrup's paper was written. By taking current regulations into account, our model may more closely reflect the current cost of compensating passengers.

Finally, the anticipated effects of the traffic factor and the safety factor on the average wait time τ suggests that our value of 2.6 hours may even be an underestimate for the present or the near future. If this is true, it is clear that the cost of compensating involuntarily bumped passengers may be higher than airlines realize, and what is more, may continue to increase. To the extent that more aggressive overbooking policies are enacted in an environment of higher waiting times between flights, the cost model we have presented here indicates that any delays caused by the combination of higher security and fewer flights could lead directly to higher costs via the implementation of the DOT compensation policy. With all of this said, involuntarily bumped passengers comprise a minority of all bumped passengers, even though the cost of handling them may be higher. In the next section, we turn to an examination of voluntary bumping costs and of possible mechanisms to deal with passengers who are willing to stand by for a price.

Voluntary Bumping: The Auction Method

The difficulties arising from bumping unwilling passengers are hard to overlook. The loss of customer goodwill and the inevitable decline in market share is further exacerbated by legal difficulties and the possibility of lawsuits. In 1968, J.L. Simon proposed a solution to these problems based on an auction. Each ticketed passenger contending for a spot on the plane would submit a sealed envelope containing the smallest sum of money for which he would give up his spot on this flight, and wait until the next available one. If there were not seats for all the contenders, the airline could then compensate the passengers who required the least money, and require that they give up their tickets. Simon argued this would be better for passengers, since they would never get bumped without suitable compensation, and it would also be better for airlines since they could safely raise their overbooking level much higher than they would otherwise be able to do. After Ralph Nader successfully sued Allegheny Airlines for bumping him, variants on this scheme have gradually become standard throughout the industry.

There are two reasonable ways to attempt an auction.

1. The first, suggested by Simon, is to force every contender to choose *a priori* a price for which they would give up their ticket. The airline would then be able to arrange all bumpings immediately.

2. The second, which is actually practiced by most airlines, is to announce possible compensation prices in discrete time intervals. Customers could then accept any offer they wished to.

The first possibility is attractive because it runs instantly, and it allows airlines to compensate each customer the absolute minimal amount of money. The second possibility, on the other hand, can cause a delay leading to customer dissatisfaction, and airlines will always pay slightly more than the minimal compensation to each bumped passenger (because only a finite number of compensations are being offered). On the other hand, an auction of the second type can be started well before a flight departs and still have enough passengers present to obtain good results. Also, if intervals are increased gradually enough, the difference in costs is negligible. Thus, these methods should generate similar results, and for simplicity, we will concentrate on the second one, although with continuous compensation offerings.

Voluntary Bumping: Continuous Time Auction Results

We now give an in-depth analysis of the results from an auction with continuously increasing compensation. In the literature, it has been common to assume that if m passengers are given compensation through an auction, the total compensation cost for the airline should be linear in m, although some authors, such as Smith et al. (1992), recognize that this function should be non-linear and convex, but do not analyze it further. In fact, it is possible to say a great deal more with only a few basic assumptions. Indeed, suppose

- n ticketholders check in for a flight with capacity C, where $n > C$ (thus, there are n contenders for C seats).

- Each contender has some minimal compensation price for which he will be willing to give up his ticket. Call this his **limit price**.

- An airline can always rebook a ticketholder on one of its own later flights at 0 cost (i.e., it does not have to pay for a ticket on a rival airline).

In an ideal auction, the airline will offer successively higher compensation prices, and whenever this exceeds a contender's limit price, he will give up his ticket voluntarily. To analyze this, we first suppose we have ticketholders $(\Gamma_1, \Gamma_2, \ldots \Gamma_n)$. For convenience, order these in such a way that Γ_i's limit price is less than Γ_j's limit price if and only if $i < j$. Define:

$D(x)$ to be the probability that a randomly selected ticketholder will give up his ticket for a price of x.

Y_m to be the amount of compensation the airline must pay to Γ_m to make him give up his ticket.

X_m to be the *total* amount of compensation the airline must pay in order to have m contenders give up their tickets.

Note that since we assume Γ_i's limit price is less than Γ_j's limit price if and only if $i < j$, $X_m = \sum_{i=1}^{m} Y_i$. Therefore, in order to determine $E[X_m]$, we need only determine $E[Y_i]$ for $i \leq m$. To do this, we need the following result:

Proposition 7.2.

$$E[Y_m] = \sum_{i=0}^{m-1} \binom{n}{i} \int_0^{\infty} \Big(D(x)\Big)^m \Big(1 - D(x)\Big)^{n-m} dx$$

The proof of this is lengthy, and not immediately relevant, so it has been placed in the appendix. Very little can be done beyond this point without further knowledge about the nature of $D(x)$. There is not much recent data on this, but when airlines were first considering moving to an auction-based system in 1978, K.V. Nagarajan polled airline passengers on their limit price. Although he performed little analysis, we found that the cumulative distribution function of this limit price fit very closely with exponential curves of the form $1 - e^{-Ax}$ for a fixed A (see Figure 4).

Although the precise figures found by Nagarajan are almost certainly out-dated, this strongly justifies the assumption that $D(x) = 1 - e^{-Ax}$ for some A, chosen independently of x. After making this assumption, we can in fact, compute $E[X_m]$ exactly. The following proposition does precisely that: (see the appendix for the proof)

Proposition 7.3. *If $D(x) = 1 - e^{-Ax}$ for some constant, A, then*

$$E[X_m] = \frac{1}{A}\left(m - (n-m)\Big(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{n-m+1}\Big)\right)$$

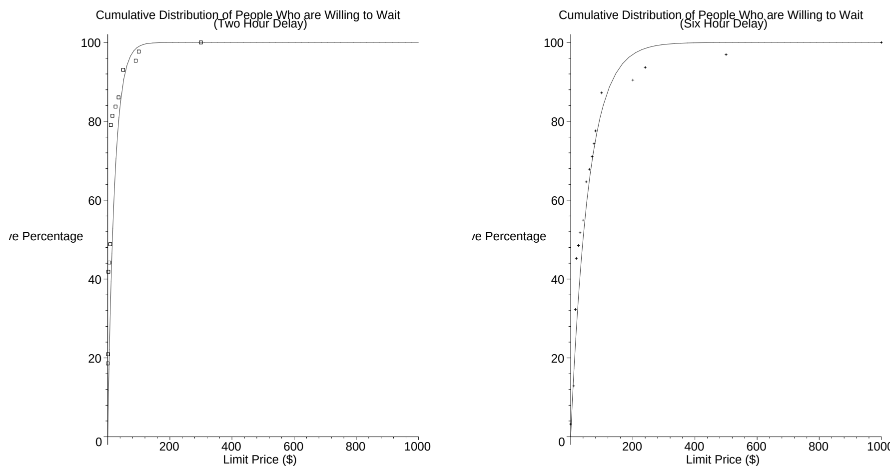


Figure 4: Polled distribution of ticketholder limit price, with best fit graphs $1-e^{0.046x}$ for the 2-hour wait and $1-e^{0.0175x}$ for the 6-hour wait [Nagarajan, 113]

Using the approximation $\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n} \approx \ln n$, we reduce this to $E[X_m] \approx \frac{1}{A}\Big(m-(n-m)\ln(\frac{n}{n-m})\Big)$. In doing so, we have taken a rather complicated scenario and simplified the expected airline cost to a closed form expression. This will prove invaluable when we construct our final model in the next section.

It remains only to consider the value of A. Unfortunately, there is no reason to believe this is constant across all scenarios. For example, contenders will certainly accept a smaller compensation for their tickets if the next flight is arriving soon. Moreover, the exact values generated by Nagarajan's results are not very meaningful due to inflation and to the fact that people will usually imagine themselves to be more flexible in a hypothetical poll question than they would be in real life. This is certainly an area where more research is necessary.

For the purpose of this study, however, we will assume A to be constant over all situations, and we will estimate that on a flight with capacity $C = 150$ and only a small number of overbooked passengers, Γ_1 will have a limit price of \$100. Thus, we have $\frac{1}{A}\cdot\frac{1}{150} \approx \$100$, so $A \approx \frac{\$1}{15{,}000}$.

In conclusion, we have shown that in an idealized auction with an exponential demand function (as suggested by Nagarajan's results), the expected amount of compensation required to bump m out of n ticketholders is approximately

$$\frac{\$1}{15,000}\left(m-(n-m)\ln\left(\frac{n}{n-m}\right)\right).$$

Summary

A great deal of time and experience has suggested that auctioning, forced bumping, and combinations thereof, are the most viable strategies for airline companies. Indeed, if a passenger is to bumped involuntarily, there is no reason to pay him much more than is absolutely necessary, since he will probably be furious regardless. Similarly, if a passenger is to voluntarily give up his ticket, the auction model will encourage him to do so at a price very close to his limit. In both scenarios, there is also a great deal of room for haggling, but without a great deal more data than is available to the public, that is very difficult to model. Since no other plausible alternatives have been suggested in the literature on this subject, it seems that these two methods of auctioning and forced bumps are truly the best available for airlines.

In this section, we have thus successfully modeled the direct compensation cost of both options. Indeed, we estimated that if m out of n contenders are bought off through an auction, the expected cost to the airline will be:

$$\frac{\$1}{15,000}\left(m-(n-m)\ln\left(\frac{n}{n-m}\right)\right)$$

On the other hand, as argued by Simon, these customers will probably be satisifed, and their business will probably not be lost. The expected cost to an airline of forcefully bumping m out of n contenders will be:

$$\$255\cdot m$$

Of course, a customer who is bumped forcefully will almost certainly be very upset, and there is a very good chance that he will try to switch to a competitor. In any case, the analysis we have done here will prove extremely important when we construct our final model in the next section.

8 Effects of Overbooking Policy on Airline Market Share: The Interactive Simulation Model

Introduction

Even in an over-simplified one-plane scenario, we were unable to solve the overbooking problem without computer assistance. When we add competition and the effects of different compensation schemes into our model, pure mathematical analysis becomes completely intractable. In order to confront this problem, we turn to a computer simulation of the airline industry and its customers.

Constructing the Model

Introduction

We will focus on the ten largest US airlines, using recent statistics to determine their flight frequency and market share. Flights will be modeled as being identical in all respects except for market interest. The market will be simulated as a group of initially 10,000 independent people, each loyal to one airline. They will buy tickets to flights offered by their airline with a fixed probability and they will meet reservations they have made with a fixed probability. Each member of the market will independently choose to stay with a company or relocate based on his treatment in each flight. To begin with, we will only consider the market prior to the events of September 11.

It is in the nature of simulations that for each factor one models properly, one has to add assumptions about a number of other smaller factors. This case is no exception. Due to the complexity of our model, we have to make a relatively large number of minor assumptions, which we now discuss:

The airline companies

Between them, the ten largest US airlines (Alaska, America West, American, Continental, Delta, Northwest, Southwest, Trans World, United, US Air) account for more than 90 percent of the domestic market, so we will focus on them. Each company will choose a number, r, specifying its overbooking strategy. This will mean that on a flight of capacity, C, the company will sell up to, but no more than, $B = Cr$ tickets. See Table 4 for the data used for each company's flight frequency and initial market share.

Airline	Market share	Flights per year
Alaska	0.0172	29,205
America West	0.0386	50,250
American	0.1668	189,149
Continental	0.0815	111,064
Delta	0.1874	222,091
Northwest	0.0947	127,791
Southwest	0.0855	122,622
Trans World	0.0499	67,344
United	0.1492	168,899
US Air	0.1292	201,497

Table 4: 1988 - 1997 Airline Average Statistics [Suzuki, 141]

Flights

In each time period, we will assume precisely one type of flight is being offered. The chance that a given airline will offer that flight is proportional to the number of flights it offers per year (see Table 4). We also determine a constant, k, indicating the world interest in this type of flight. Beyond that, different types of flights are assumed to be identical. In particular, each flight will have capacity $C = 150$ seats each sold at \$140. We will also assume that flights have only one type of fare. (See section 6.4 for a justification of the fact that multiple fare classes will have negligible effect upon the optimal overbooking strategy).

The market

The exact size of the market should have little effect on the result, so long as we ensure that the average load factor is consistent with reality. Thus, we will assume that the total market is initially made up of 10,000 independent people, each loyal to one carrier. The relative sizes of the company market shares will be initialized according to industry data prior to September 11 (see Table 4). We will also assume each person in the market will, on average, fly the same number of times in a year.

Market interest in individual flights

If a company sets an upper limit of $B = Cr$ tickets to be sold for a given flight, there is certainly no guarantee that it will actually sell that many, as we assumed would happen in the One Plane Model. Instead, we will assume here that each person in a company's market has probability k of wanting to buy a ticket for the flight (here, k refers to the world interest described above). It thus remains only to determine k. We will use a normal distribution, and we will fix its mean in such a way that the average load factor on all flights is the industry average of 0.72 [BTS website, www.bts.gov/].

Single flight simulation

We used industry data from prior to September 11 to estimate that the probability each ticketholder will check in for his flight is 0.85. If necessary, each airline then bumps some passengers voluntarily and some involuntarily, according to whatever strategy it is using. The immediate cost of bumpings will be set to the values that we derived in the previous section. We estimate that voluntarily bumped passengers will be relatively happy, and thus will leave the airline with probability only 0.05, whereas involuntarily bumped passengers will be furious, and will leave the airline with probability 0.8.

Market change

If a person leaves an airline, we assume he stays within the market with probability 0.9 (or 0.95 if he was bumped voluntarily). In this case, he simply switches to another company; otherwise, he leaves the market altogether. Wo also assume that people trickle into the market fast enough to compensate for the loss of people due to dissatisfication, thus allowing the market to slowly grow.

Summary

We conclude with a quick overview of the simulation model:

- There are 10 airline carriers starting with a total of 10,000 customers
- Each time period, one flight type is available. The probability that any given carrier will offer that flight is proportional to its real world flight frequency.
- Each person in the company's market decides independently whether or not to buy a ticket for each flight.
- Each ticketholder checks in with probability 0.85.
- The airline compensates bumped passengers according to whatever combination of voluntary and involuntary bumping it chooses.

- Customers individually decide whether to remain with the current airline after each flight, or whether to switch to a different airline or out of the market altogether.

The source code for the simulation has been included in the appendix for reference.

Simulation Results, pre-September 11

We first investigated the effect that different overbooking rates would have on profit using our pre-September 11 model. Then, for each possible overbooking rate that this one company could choose, we calculated its net profit over 500 time periods (ensuring that the same random events occured regardless of the strategy being tested). The strategy that maximized profit for that time period was then determined and tabulated. This was repeated 40 times for each company.

This leaves open the question of what strategies the companies not being tested should use. To determine this, we first assumed each company would overbook by 1.17 (as computed in the single-plane model), and then ran the program to get a first estimate of what a good strategy might be. Assuming that companies would tend to gravitate towards good strategies, we then used the optimal results from that preliminary run to set the default overbooking rates of each company in the final run. Finally, we used the industry figure that 5% of all bumped passengers are bumped involuntarily to set the company compensation strategies.

The results generated by the simulation are shown below in Figure 5, and in more detail in Figure 6.

Analysis

Ideally, we might expect the optimal overbooking rate for each company to either be constant, or perhaps to be distributed normally. Either way, for the results of the simulation to be meaningful, it is extremely important that the program generate similar answers each run. Looking at the histograms shown in Figure 6, it is clear that this fails completely for Alaska. This is, however, not unreasonable. We can check in Table 4 that Alaska has far fewer passengers per flight than its competitors, and as a result, it rarely fills any plane entirely. As such, its overbooking policy will have a neglible effect on its overall profit. Thus, the simulation will almost certainly be too coarse to generate useful data. For the other nine companies, however, the simulation was extremely consistent.

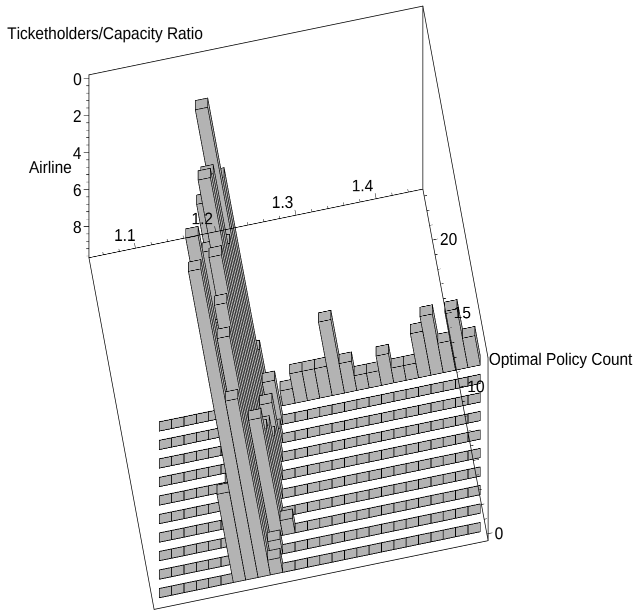
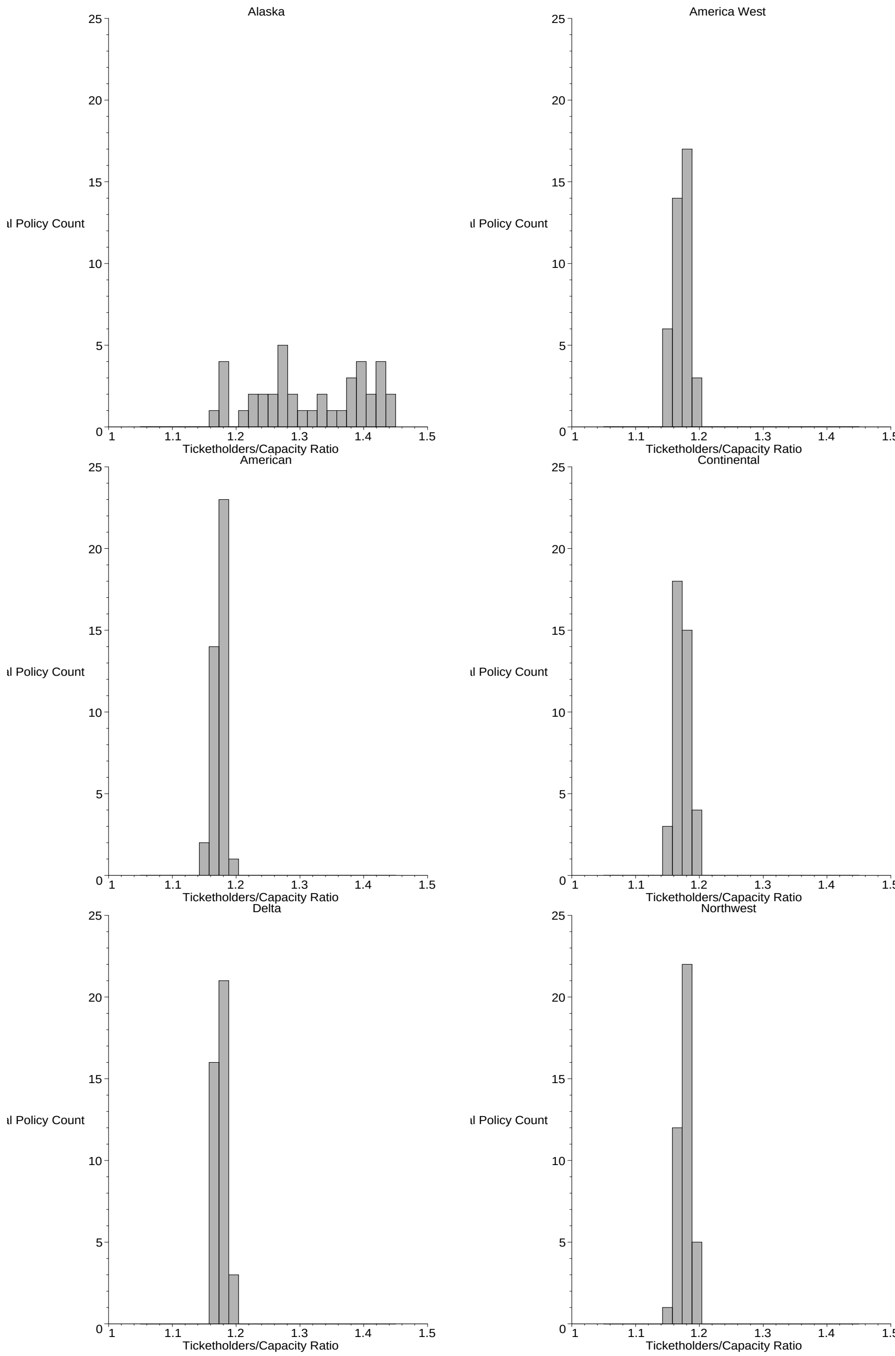


Figure 5: Simulation Results - The number of runs (out of 40) for which each overbooking rate proved optimal

Airline	Optimal overbooking rate over 40 runs
Alaska	1.319
America West	1.169
American	1.171
Continental	1.170
Delta	1.173
Northwest	1.174
Southwest	1.173
Trans World	1.176
United	1.168
US Air	1.165

Table 5: Compiled simulation results

Also, looking at the average results generated over all 40 runs for each airline, we notice that the optimal overbooking rate for all companies other than Alaska is between 1.165 and 1.176 (see Table 5), which is close to,


Alaska
America West
American
Continental
Delta
Northwest
l Policy Count
Ticketholders/Capacity Ratio
0
5
10
15
20
25
1
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1.2
1.3
1.4
1.5

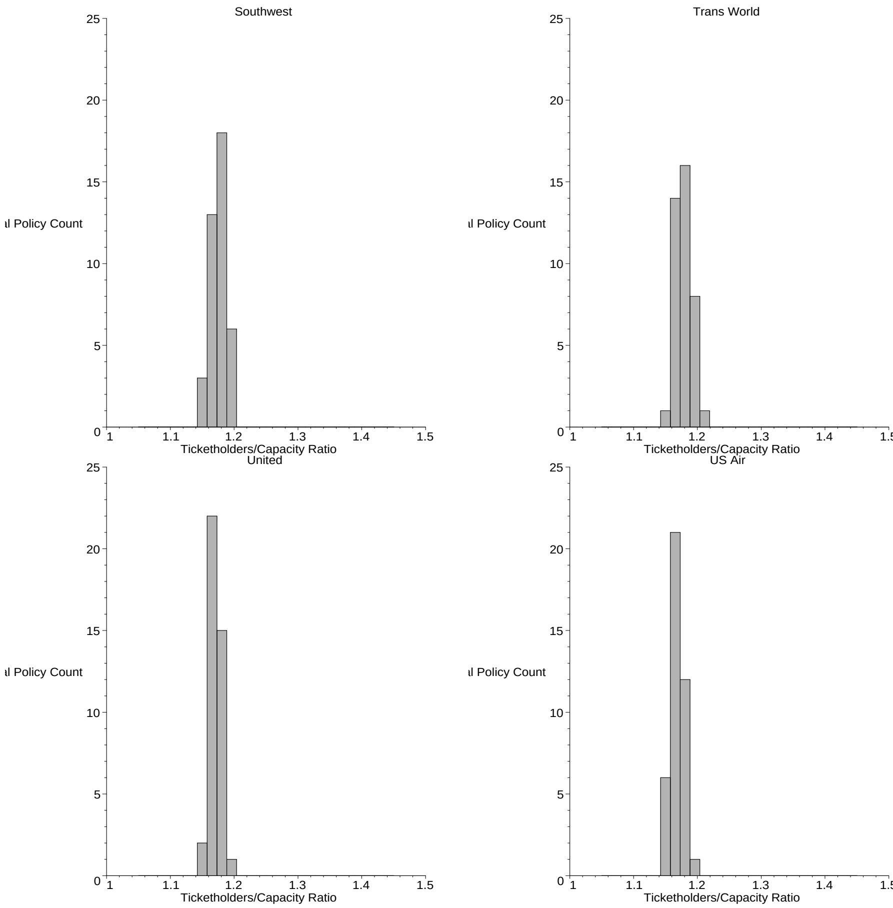

Figure 6: Simulation Results - The number of runs (out of 40) for which each overbooking rate proved optimal

although a little less than, our results from the One Plane model. This is reasonable, since the most significant improvement that this simulation makes over the One Plane Model is the consideration of lost customers, an effect which should slightly reduce the optimal overbooking rate. Thus, it seems reasonable to assume that although the results for Alaska are unreliable, we have genuinely accurate data for the other nine carriers.

Adjusting the Model due to September 11

We now set up another run of our Interactive Simulation Model to determine optimal overbooking strategy for the post-September 11 state of affairs. To do so, we estimate what effect the complicating factors will have on the simulation parameters:

1. *Arrival Probablity p: increases from* 0.85 *to* 0.90. The security factor makes it difficult for passengers to arrive at their flights on time due to long security checks. Because of the fear factor, however, a large proportion of post-September 11 passengers fly only out of necessity, and hence are much less likely to risk missing their flights. This suggests a positive net change.

2. *Flight Frequency: decreases by* 20%. This is a direct consequence of the traffic factor. Parker, a growth airline analyst at Raymond James, estimates the decrease in flight frequency to be approximately 20% [www.raymondjames.com].

3. *Total Market Size: decreases by 15%.* Total market size has almost certainly decreased post-September 11 due to the fear factor. However, fourth quarter data from 2001 is not yet available, so we are forced to make an estimate. It has been our experience that flights are more crowded now than they were, which suggests the percentage of market size decrease has been smaller than the percentage of flight frequency decrease. Thus, we estimate the market size has decreased by 15%.

4. *Market Return Rate: doubles.* The market size has decreased due to the fear factor, but Parker [www.raymondjames.com] anticipates that demand will return to pre-September 11 levels by mid-2002. Moreover, public perception of airline safety is improving due to the security factor. Thus, the market return rate should be substantially higher than its pre-September 11 level.

5. *Market Exit Rate: decreases by* 50%. As mentioned above, the market composition is now more heavily weighted towards those who fly only out of necessity. Such fliers are much less likely than casual fliers to leave the market altogether after a bad bumping experience.

6. *Percentage of Bumps which are Voluntary: decreases from* 95% *to* 90%. Due to the traffic factor, there are fewer flights, and hence the waiting time between flights will increase. Moreover, since passengers are more

likely to be flying of necessity, they will be much less interested in giving up their ticket for financial compensation.

7. *Compensation Cost of Voluntary Bumping: increases by* 20%. As mentioned above, contenders will be less willing to give up their tickets, and hence, will only do so if they receive a higher compensation.

8. *Compensation Cost of Involuntary Bumping: increases by* 20%. As mentioned above, bumped passengers will face longer waiting times before they can catch an alternate flight. By virtue of DOT regulations, this means that involuntary compensation costs will rise.

9. *Competitor Agression: increase in competitor overbooking levels from* r *to* $r+0.02$. Due to the financial loss factor, any given airline can expect its competitors to focus more heavily on short-term profits than they would have previously. Unless these airlines perform a very careful analysis, they will probably assume this can be best accomplished by choosing more aggressive overbooking stragies.

Simulation Results, post-September 11

Using the parameter changes outlined in the previous section, we ran the simulation again to estimate the effect that the events of September 11 have had on optimal overbooking strategies. The results are shown in Table 6.

Airline	Pre-September 11 optimal overbooking rate	Post-September 11 optimal overbooking rate
Alaska	1.319	1.260
America West	1.169	1.094
American	1.171	1.094
Continental	1.170	1.096
Delta	1.173	1.095
Northwest	1.174	1.095
Southwest	1.173	1.095
Trans World	1.176	1.096
United	1.168	1.094
US Air	1.165	1.092

Table 6: Compiled simulation results

Although we did not include it here due to space considerations, we can construct histograms similar to those in Figure 6, and observe that once again, the data is extremely consistent for all companies other than Alaska. Moreover, there is again a strong correllation between the simulation results for these parameters and the corresponding results from the One Flight Model. This agreement reflects well on the analysis done with both.

Moreover, from Table 6, it is clear that the events of September 11 have indeed had a significant effect on optimal overbooking rates. Indeed, for a company the size of American Airlines, a 7% change in these rates could easily lead to a difference in profits on the order of 1 billion dollars.

Thus, if our estimates of parameter changes due to September 11 were reasonable, all major airlines should significantly decrease their overbooking rates. In any case, any companies that have not seriously reevaluated their overbooking policy to account for recent market changes should do so now, possibly improving upon our own model by using the data they have that we do not.

9 Conclusion

In this paper we developed several models to examine the effects of different overbooking policies on airline revenue and costs in light of changes in the state of the industry due to September 11. In the past several months, airlines have experienced these four complicating factors: decreased flight volume, increased security, heightened passenger fear, and billions of dollars worth of financial loss. We modeled the situation both before and after September 11 and were able to make policy suggestions based on our models.

We began by considering the single plane case of the overbooking problem with our One Plane Model. Although this model was somewhat simplistic, it produced reasonable suggestions for overbooking policy. In particular, our computer program was able to determine the optimal overbooking policy when the plane capacity, arrival probability of ticketholders, and the compensation to bumped passengers were specified. We also used the One Plane Model to make some prelimary predictions as to how airlines should alter their overbooking policy in light of the recent industry changes.

The Extended One Plane Model considered the single plane case with multiple fare classes: first-class and coach. Like the basic One Plane Model, this model produced very reasonable suggestions for overbooking policy. We analyzed the results of this model and determined that multiple fare classes had little to no effect on the optimal overbooking policy. Thus, we chose not to consider multiple fares in our more complicated Interactive Simulation Model.

Next, we considered alternative compensation schemes for dealing with passengers who are bumped due to overbooking. Department of Transportation regulations require that passengers bumped involuntarily be compensated according as to how long they must wait for the next available flight. We modeled this using an exponential waiting time distribution, and found the expected amount an airline should expect to have to pay to involuntarily bumped passengers. We also extensively modeled compensation schemes for voluntarily bumped passengers. In particular, we developed an auction model which closely fits the real-world scheme airlines actually employ. As for the question of the best compensation method, we conclude that it is a combination of auctioning and forced bumps which we spelled out in detail.

The most important part of our paper was the Interactive Simulation Model. In it, we expanded very substantially upon the One Plane Model. We included a host of factors, including multiple flights over an extended time period, multiple competing airlines each with different overbooking policy, and individuals switching to competitors due to being bumped. We ran the simulation for a large number of trials to obtain more reliable results. We obtained optimal overbooking policy suggestions for each of the 10 largest US carriers.

We carefully analyzed how the four complicating factors due to the events of September 11 would change the inputs in our simulation. We then changed these input parameters and re-ran the simulation in order to determine appropriate changes for airlines' overbooking policies post-September 11. From the results of our simulation, we concluded that the optimal policy for airlines is to decrease their overbooking. Moreover, with certain data that we lack but that airlines surely have, airlines can determine very precisely what new their optimal overbooking policy should be.

We note that this problem is of great importance to airlines, since small deviations from the optimal overbooking strategy can easily result in huge financial losses in fairly short order, as our analysis shows. Airlines can insert certain empirical data they have collected into our models in order to make accurate determinations of optimal overbooking policy. In conclusion, we recommend our models to airlines.

10 Memorandum

Attn: Don Carty, CEO American Airlines
From: MCM Team 180
Subject: Overbooking Policy Assessment Results

We completed the preliminary assessment of overbooking policies that you requested. There is a great deal of money at stake here, both from ticket sales and also from compensation that must be given to bumped passengers. Moreover, if too many passengers are bumped, then there will be a loss of goodwill and many regular customers could be lost to rival airlines. In fact, we found that the profit difference for American Airlines between a good policy and a bad policy could easily be on the order of 1 billion dollars a year.

Using a combination of mathematical models and computer simulations, we considered a wide variety of possible strategies that could be tred to confront this problem. We naturally considered different levels of overbooking, but we also looked at different ways in which airlines could compensate bumped passengers. In terms of the second question, we found that the current scheme of auctioning off compensations for tickets, combined with certain calculated forced bumpings, is still ideal, regardless of changes to the market state.

Although we were forced to work without much recent data, we were also able to achieve reliable and consistent results for the optimal overbooking rate. In particular, we found that prior to September 11, American Airlines stood to maximize profits by selling approximately 1.171 times as many tickets as there were seats available.

We next considered how this number would likely be affected by the current state of the market. In particular, we focused on four consequences of the events on September 11: all airlines are offering less flights, there is heightened security in and around airports, passengers are afraid to fly, and the industry has already lost billions of dollars in revenue. Analyzing each of these in turn, we found that they did indeed have a significant effect on the market. In particular, American Airlines should lower its overbooking rate to 1.094 tickets per available seat.

In conclusion, we found that there is indeed a tremendous need to reevaluate the current overbooking policy. According to our current data, we believe that the rate should be dropped significantly. It would be valuable, however, to supplement our calculations with some of the confidential data that American Airlines has access to, but that we do not.

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11 Appendix A: Proofs

Proposition 11.1. *Consider the scenario presented in section 7.4. Then,*

$$E[Y_m] = \sum_{i=0}^{m-1} \binom{n}{i} \int_0^{\infty} \Big(D(x)\Big)^m \Big(1 - D(x)\Big)^{n-m} dx$$

Proof. Let $G_m(x)$ be the probability that precisely m contenders will give up their tickets voluntarily for a compensation of x. Then, since the reactions of the n contenders are all independent,

$$G_m(x) = \binom{n}{m}\Big(F(x)\Big)^m\Big(1 - F(x)\Big)^{n-m}$$

Let $D_m(x)$ be the probability that Γ_m has a limit price less than x (ie., $D_m(x)$ is the probability that a ticketholder has a limit price less than x, *given* that he has the m'th smallest limit price among all n ticketholders). Since at least m contenders will give up their tickets voluntarily for a compensation of x if and only if Γ_m's limit price is less than x, we have

$$\begin{aligned} D_m(x) &= \sum_{i=m} nG_i(x) \\ &= \sum_{i=m}^{n} \binom{n}{i}\Big(D(x)\Big)^i\Big(1 - D(x)\Big)^{n-i} \\ &= 1 - \sum_{i=0}^{m-1} \binom{n}{i}\Big(D(x)\Big)^i\Big(1 - D(x)\Big)^{n-i} \end{aligned}$$

since $\sum_{i=0}^{n}\binom{n}{i}\Big(D(x)\Big)^i\Big(1-D(x)\Big)^{n-i} = \Big(D(x)+1-D(x)\Big)^n = 1$. However, $D_m(x)$ is, by definition, the cumulative distribution function for Y_m, so

$$\begin{aligned} E[Y_m] &= \int_0^\infty \Big(1 - D_m(x)\Big)dx \\ &= \int_0^\infty \sum_{i=0}^{m-1} \binom{n}{i}\Big(D(x)\Big)^i\Big(1 - D(x)\Big)^{n-i} dx \\ &= \sum_{i=0}^{m-1} \binom{n}{i} \int_0^\infty \Big(D(x)\Big)^i\Big(1 - D(x)\Big)^{n-i} dx \end{aligned}$$

□

Proposition 11.2. *Consider the scenario presented in section 7.4. If* $D(x) = 1 - e^{-Ax}$ *for some constant, A, then*

$$E[X_m] = \frac{1}{A}\left(m - (n-m)\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{n-m+1}\right)\right)$$

Proof. Letting $u = D(x) = 1 - e^{-Ax}$, we have

$$\begin{aligned} \frac{du}{dx} &= Ae^{-Ax} \\ &= A(1-u) \end{aligned}$$

so $\frac{dx}{du} = \frac{1}{A(1-u)}$. Substituting this into the result from proposition ????, we obtain

$$\begin{aligned} E[Y_m] &= \sum_{i=0}^{m-1} \binom{n}{i} \int_0^1 u^m (1-u)^{n-m} \cdot \frac{1}{A(1-u)} du \\ &= \frac{1}{A} \sum_{i=0}^{m-1} \binom{n}{i} \int_0^1 u^m (1-u)^{n-m-1} du \end{aligned}$$

Now, $\int_0^1 u^m (1-u)^{n-m-1} du$ is a Beta function, and it is well-known that this is equal to

$$\frac{i!(n-m-1)!}{(n-m+i)!} = \frac{1}{(n-m)\binom{n}{m}}$$

Therefore,

$$\begin{aligned} E[Y_m] &= \frac{1}{A} \sum_{i=0}^{m-1} \binom{n}{i} \frac{1}{(n-i)\binom{n}{i}} \\ &= \frac{1}{A} \sum_{i=0}^{m-1} \frac{1}{n-i} \end{aligned}$$

Finally,

$$
\begin{aligned}
E[X_m] &= E\Big[\sum_{i=1}^{m} Y_i\Big] \\
&= \sum_{i=1}^{m} E[Y_i] \\
&= \frac{1}{A}\sum_{i=1}^{m}\sum_{j=0}^{i-1}\frac{1}{n-j} \\
&= \frac{1}{A}\sum_{j=0}^{m-1}\sum_{i=j+1}^{m}\frac{1}{n-j} \\
&= \frac{1}{A}\sum_{j=0}^{m-1}\frac{m-j}{n-j} \\
&= \frac{1}{A}\Big(m-\sum_{j=0}^{m-1}\frac{n-m}{n-j}\Big) \\
&= \frac{1}{A}\left(m-(n-m)\Big(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\ldots+\frac{1}{n-m+1}\Big)\right)
\end{aligned}
$$

□