Let $\mathcal{S}_n$ be the group of permutations $\pi$ of the set $\{1, 2, \ldots, n\}$.

For $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we say that $\pi(i_1), \ldots, \pi(i_k)$ is an increasing subsequence of length $k$ in $\pi$ if $\pi(i_1) < \cdots < \pi(i_k)$.

Let $\ell_n(\pi)$ be the length of the longest increasing subsequence of $\pi$.

For example, $N=5$, $\pi=13245$, $\ell_n(N, \pi) = 3$.

Equip $\mathcal{S}_n$ with the uniform distribution so that

$$p_{n,n}(\pi) = \text{Prob} \left\{ \ell_n(\pi) \leq n \right\} = \frac{\rho_{n,n}}{n!}$$

where $\rho_{n,n} = \# \text{ of } \pi$'s with $\ell_n(\pi) \leq n$.

The aim is to determine the asymptotics of $p_{n,n}$ as $n \to \infty$. 
Why is \( l_n \) of interest?

Connections to

(i) representation - theory of \( S_n \). Young tableaux (we'll later)

(ii) \( N - l_n(\pi) = \min N \) of mutation/detection steps needed to go from the identity (a library problem).

(iii) \( d(\pi, \sigma) = N - l_n(\pi \sigma^{-1}) \) Ullan's metric on \( S \) - useful for a variety of statistical questions

(iii) DNA sequencing

(iv) \( l_n \) is believed to be a particular universality class for some statistical mechanical particle models

(v) patience sorting algorithm

(vi) Some history

The beginning of the business.

(1935) Erdos + Szekeres: proved a Ramsey type theorem (from the book)}
Independent proofs were of Ulam's problem shown new by

1995 Aldous & Diaconis

1996 Seppäläinen

1997 Johansson

Over the years many conjectures have been made
about the statistics for $E_n$. In particular, 11
were various conjectures for $\text{Var}(E_n)$ of the form

$$\text{Var}(E_n) \sim c_1 N^x$$

for different values of $x$.

Then in typical fashion, together with Eric Basri

Andrew Odlyzko and I began a series of

large-scale Monte Carlo simulations on $\text{Var}(E_n)$.

Indeed:

$$\frac{\text{Var}(E_n)}{N^{1.3}} \rightarrow c_0 \sim 0.8143$$

$$\frac{\left|E(E_n) - 2N\right|}{N^{1.6}} \rightarrow c_1 \sim -1.23887$$

And that is within the usual realm.
In order to state our results on the prob.

I need to introduce some notation. Let 

\[ u(x) \]

be the (unique) solution of the Painlevé (PII) equation

\[ L u = 2u^3 + xu, \quad u(0) = A, |x| \to \infty \]

(Hurwitz, 1885; Riemann 1879)

We have (I will say more about this later)

\[ u(x) = A \ln x + O \left( \frac{e^{-(\log x)^{3/2}}}{x^{1/4}} \right), \quad x \to -\infty \]

\[ = -\sqrt{\frac{-x}{2}} \left( 1 + O \left( \frac{1}{x} \right) \right) \quad \text{as} \quad x \to -\infty \]

Set

\[ F(t) = e^{-\int_{-t}^{0} u^2(x) \, dx} \quad \text{as} \quad t \to +\infty \]

\[ \approx e^{-t \log t^3} = e^{-3 \log t} \approx 1 - e^{-3 \log t} \quad \text{as} \quad t \to +\infty \]

Then \( F(1) > 0 \), \( F(1) \to 0 \) \( t \to +\infty \), \( F(t) \to 1 \) \( t \to -\infty \)

\( F(t) \) is a distribution function.

**Theorem:** Let \( \pi \) be the cycle group of size \( N \) with an

\[ \pi \]

and \( \pi \in \text{Sym}(N) \) be the length of the longest increasing

\[ \pi \]

path \( \pi \). Let \( X^{(1)} \) be a random variable whose

distribution is \( \pi \). Then as \( N \to \infty \)
\[ X_N = \frac{\bar{X} - 2\sqrt{N}}{N^{1/6}} \rightarrow \chi \text{ in distr.} \]

\[ \lim_{N \to \infty} \text{Prob} \left( \frac{X_N - t}{N^{1/6}} \right) = F(t) \quad \forall t \in \mathbb{R} \]

We also have convergence in distribution.

Theorem 2: For any \( m = 1, 2, 3, \ldots \), we have

\[ \lim_{N \to \infty} E \left( X_N^m \right) = E \left( X^m \right) \]

where \( E \) denotes exp. and \( F \). In particular, for

\[ \lim_{N \to \infty} \frac{\text{Var}(X_N)}{N^{1/3}} = \int t^2 dF(t) \]

and for \( m = 1, 2, 3 \)

\[ \lim_{N \to \infty} \frac{\text{E}_m(X_N) - 2\sqrt{N}}{N^{1/6}} = \int t dF(t) \]

If one plots \( P \) numerically, it compares to \( \Phi(t) \) at \( t \approx 4.8 \).

(a) (b) we find \( -0.813 \) for \( t = -1.777 \), which
Now it turns out that there is a very interesting connection between the above results and another subject in a very different area, random matrices. In this theory, in particular (see [12]), one considers \( n \times n \) Hermitian matrices \( M = (M_{ij}) \) with probability density \( \mathcal{CUE} \):

\[
Z_n^{-1} e^{-\operatorname{Tr} M^2} \, dM = Z_n^{-1} e^{-\operatorname{Tr} M^2} \prod_{i,j} dM_{ij} \quad \text{for the matrix elements (i,j) of size } n.
\]

Now the fact is that \( \lambda_n / (\operatorname{Tr} e^M) \to 0 \) as \( n \to \infty \), the distribution of the

largest eigenvalue of a \( n \times n \) matrix \( M \), suitably centred and scaled \( (\lambda_n / n^{1/2}, \cdots, n^{1/2}) \), converges in distribution to the same function \( F(t) \) for our theorem says that:

the length of the largest increasing subsequence of a

where, like the largest eigenvalue of a random matrix,
Natural Question: Tracy and White also computed the distribution function for the 2nd, 3rd, ..., largest eigenvalues. Do they have anything in the same permutation picture that behaves like the 2nd (or 3rd, etc.) largest eigenvalue of a random GUE matrix?

To answer this question:

It clearly cannot be the 2nd largest eigenvalue, which is clearly distributed in the same way as all others.

To see what else to do, we must introduce some of ideas from combinatorics and representation theory.

Let $n = (n_1, n_2, \ldots, n_r)$, $n_1 \geq n_2 \geq \ldots \geq n_r$ be a partition of $\ell$.

\[
\sum_{\Pi} = 1
\]

Associate to $\Pi$ an integer $x_1 \geq x_2 \geq \ldots \geq x_r \geq 0$ with $x_1 + \cdots + x_r = \ell$.
If we insert the \( n \) 's \( 1, \ldots, n \) (write -1 in boxes of the \( n \) frame) we obtain an \( n \times n \) Young tableau.

\[
\begin{array}{ccc}
1 & 4 & 5 \\
6 & 3 & 2 \\
\end{array}
\]

of \( n = 6 \).

If we ensure that all rows and columns are increasing

\[
\begin{array}{ccc}
1 & 2 & 6 \\
3 & 4 & 5 \\
\end{array}
\qquad
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 & 6 \\
\end{array}
\]

we obtain a \underline{Standard Young Tableau} which may be \underline{matched}.

Now there is a remarkable \underline{Theorem of Robinson (1938) and Schensted (1961)} which says there is a \underline{bijection}

\[
S_n \leftrightarrow \pi \quad \text{and} \quad (P(\pi), Q(\pi))
\]

\( \leftrightarrow \)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}
\]

from \( S_n \) onto parts of a Young tableau with \( \pi \) rows.

\[
S_n \leftrightarrow \pi \quad \text{and} \quad (P(\pi), Q(\pi))
\]

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}
\]

\[
S_n(\pi) = \mathcal{A}_n (P(\pi))
\]

Theorem of Robinson (1938) and Schensted (1961)
tables below should be filled with random numbers for the same reason.

...what about the 2nd row of boxes in the *first* column from (\(p(n)\)) to (\(q(n)\))?

Well, if we go back to the simulation of Odlyzko and bring up the full list of the computations, we have:

\[
\lim_{N \to \infty} \frac{\mu_2(n)}{N^{1/6}} = 0.546
\]

\[
\lim_{N \to \infty} \left( \frac{\mu_2(n)}{N^{1/6}} - 2\sqrt{N} \right) = -3.618
\]

Well, those values agree, once again, with the variance on the mean of the 2nd largest eigenvalue (mutually centered and scaled) of a random GUE matrix as computed by Tracy and Widom! So...
The following reason: in the early '60s

Yin Zhou and I introduced a stepwise descent type

oscillating method by setting off problems with

as some external parameter. This work was developed

by a lot of people and eventually placed in a general form

by R. Zhou, Venkataram and myself. The method is a non-iterative, non-linear learning of the classical

Steepest descent method for integrals. I'll say more

about this latter: also tomorrow will. But in no

way different in any different case, thus reducing to the asymptotic solution of

Step 1: Poisson Problem

Set

\[ \Phi_n^{(1)}(\lambda) = \sum_{N=0}^{\infty} \frac{e^{-\lambda} \lambda^N}{N!} q_{n,N} \]
By a rather elementary, but fortunate, use of Poissonization property
(Enlarged Theorem, §1.6.3.4) the theorem,

\[ \text{asymptote of } \phi_n(x) \text{ as } n \to \infty, \]

can be inferred from

The asymptotes of \( \phi_n(x) \) for \( n \to \infty \)

So we must investigate the double scaling limit for

\[ \phi_n^*(x), \quad \text{for } x \in N \times \]

Poissonization helps because of the following, convenient for

There is an explicit formula for \( \phi_n^*(x) \):

\[ \phi_n^*(x) = e^{-x} D_{n-1}(e^{2i \pi x}) = e^{-x} \delta_{n-1}(x) \]

\( D_{n-1} \) is the non-trivial determinant with weight function

\[ f(e^{i \theta}) = e^{2i \pi \theta} \]

\[ D_n(x) = \det \left( \theta_{ij} - f(e^{i \theta}) \frac{\partial}{\partial \theta} \right) \]

This formula was first found by Gessel (1990) but later
never been discovered independently by many authors.
Step 2

Set \( k_n^+(x) = \frac{D_n(x)}{D_\infty(x)} \)

\( k_n^+(x) \) is the normalized coefficient by the nth
orthogonal polynomial \( p_n(x) \):

\( p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2 - 1, \ldots \)

\[
\int_{-\infty}^{\infty} p_n(e^{i\theta}) p_m(e^{i\theta}) \frac{d(e^{i\theta})}{2\pi} = \delta_{nm}
\]

Using the Stieltjes-Cebysev limit theorem we have

\[
\log d_n(x) = \sum_{k=1}^{\infty} \log k_n^+(x)
\]

This is the main formula to compute \( d_n(x) \)

we must control \( k_n^+(x) \) for \( (k,x,n) < \infty \)

Steps 3

Nielsen-Hilbert projection

What is a NHP?

Suppose we have an oriented contour \( \Sigma \) in \( \mathbb{C} \)
By convention, if we traverse an arc in \( \mathbb{C} \mathbb{S} \) in

the direction of the arrow, we say that the + (resp. -)

side lies to the left (resp. right).

Suppose we are given a (smooth) map \( \nu: \mathbb{S} \rightarrow \mathbb{C} \).

Then the Dirichlet problem \((\Sigma, \nu^-)\) is the following:

1. \( m = m(\nu) \) such that

\[ m \] is analytic in \( \mathbb{C} \setminus \Sigma \)

2. \( m(z) = m(z) \nu(z) \), \( z \in \Sigma \),

where \( m(z) \) is defined on \( m(z) \).

If in addition

\[ m(z) \rightarrow I \] as \( r \rightarrow 0 \),

then the pair \((\Sigma, \nu^-)\) is normalized at \( 0 \)
For us the facts of the matter are as follows:

(Continued from previous text):

Let $\Sigma = \omega$ and consider counterclockwise:

Let $\gamma(z; k+1, 2)$ be the matrix function satisfying the following RHP:

- $\gamma(z; k+1, 2)$ is analytic in $\mathbb{C} \setminus \Sigma$
- $\gamma(z; k+1, 2) \sim \gamma(z; k+1, 2) \begin{pmatrix} 1 & \frac{1}{2k+1} e^{i\pi (3+\frac{1}{2})} \\ 0 & 1 \end{pmatrix}$
- $\gamma(z; k+1, 2) \begin{pmatrix} z^{-(k+1)} & 0 \\ 0 & z^{k+1} \end{pmatrix} = I + O(\frac{1}{z}) \quad \text{as} \quad z \to \infty$

Then $\gamma$ is unique and

$$ \kappa_k^2(z) = -\gamma_{21} (\omega^2; k+1, 2) $$

Also $\frac{\kappa_{k+1}}{\kappa_k} = \frac{\gamma_{21}}{\gamma_{11}}$

So to evaluate $\kappa_k^2(z; k+1, 2)$ and hence $f_{m1}(z)$,

we must be able to construct the solution $\gamma$.
The above LHP in the limit when the "excitation" parameter \( \delta \) in \( \zeta = (\omega - i \mu) e^{\pm i \gamma} \) are large.

This is precisely the situation that can be corrected by the non-linear steepest descent method. The calculations are very similar to those that are in the works of D. McLay, Krichever, Venakides, & the presentation of universality of various statistical quantities in random matrix theory.

\[ \text{Signature: } \text{Professor} \]

\[ \text{Remark: The above LHP for } \Gamma \text{ is strongly related to the LHP's that arise in the analysis of the Toda lattice equation and the Painlevé III equation.} \]
That $\left< \text{since } \frac{p}{q} \right>$ must satisfy some identities, what the
one, we do not yet know, but this seems
to be a way to get new identities for relevant
combinatorial quantities.

Step 4. Rainever There.

What does $P_{12}$ come into this picture?

Now there is a wonderful way (very hard essential
for Alhazid's paper, then Fadlina - Newell & our first,
the to solve that

$$n^* = 2n^3 + k_n$$

Let $p, q, r$ be 3 #s satisfying $p + q + r = 0$

Consider $\mathcal{C}$ consisting of 6 rays with $\mathcal{N}$ attached as
follows:
\[ (0, 0) \quad \text{and} \quad (0, \beta) \]

Let \( N \) denote the following set:

\[ N \equiv \{(x, y) \in \mathbb{R}^2 : x + y = 0\} \]

Then \( \psi = \psi_N \) and \( \Sigma = \mathbb{R}^2 \setminus N \).

Also, \( \psi_N = \psi_N^{-1} \).

Moreover, \( \psi_N = \psi_{(a, b)} \).

Then \( q = m \).

\[ m = 2 \pm \frac{m(x, y)}{3} + O \left( \frac{1}{\delta} \right) \]

\[ u(x, y) = 2 \pm (m_x, m_y) \]
Of particular interest is the case where

\[ p = \frac{1}{2}, \quad r = \frac{1}{2} \]

which can be deduced to a problem of the form:

\[ (1, 0) \quad \rightarrow \quad (0, 1) \]

\[ (0, 1) \quad \rightarrow \quad (1, 0) \]

Now it turns out, at least for \( \frac{1}{2(n+1)} \),

the RHP for \( Y \) is equivalent to a RHP on a contour of the following form

\[ \text{Area: } L \]
clearly noticable. Then come for $3 \rightarrow 1$.

\[ \text{Notable by } 90^\circ \text{ or } 45^\circ \]

which is the content precisely. The rest, move the right way jump variates for $P_\Pi$.

In this way $P_\Pi$ comes into $\Pi$ the picture.

Analytical remark: what kind of an analytic problem is this really.

Homogeneous point $m = (1 - C_{ij})^{-1} I$.