Gergen Lectures . Duke University

October 22-24, 2001

Random Matrix Theory

Lecture 1

It is a great pleasure and honor for me to give the year 2001 Gergen Lectures I am also very pleased to be here with so many of my friends. My topic is random matrix theory with emphasis on the relationship to integrable systems

In this the first of 4 lectures, I am going to begin with some

- general remarks about integrable systems.

Then I will

- make some general and historical remarks about random matrix theory (RMT)

- universality conjecture in RMT

In the $\boxed{2^{nd}}$ lecture, I will
- introduce a variety of objects and techniques from modern theory of integrable systems

In the $\boxed{3^{rd}}$ lecture, I will show how
- to use these techniques to solve the universality conje following the approach of T. Kriecherbauer, K. McLaughli S. Venakides, X. Zhou and P.D.

In the $\boxed{4^{th}}$ and final lecture, I will show how

- ~~to~~ these these techniques to analyze the so-called Ulam problem in the theory of random permutations.

Integrable systems

The modern theory of integrable systems began in with the discovery of Gardner, Greene, Kruskal and Miura of a method for integrating the Korteweg de V

(KdV) equation

$$u_t + uu_x + u_{xxx} = 0 \qquad , \; x \in \mathbb{R}, \; t \geq$$
$$u(x,t=0) \to 0 \quad \text{as} \quad |x| \to \infty$$

This equation arises in the theory of water waves

Initially the discovery of Gardner et al was regarded as providing a method of solution for a rather thin set of evolutionary equations, but by the early 1980's it started to become clear that the discovery of Gardner et al was just the first glimp of a far more general integrable method that woul eventually have applications across the broad spectrum of problems in pure and applied mathematics. In the narrowest sense, an integrable system is a

Hamiltonian dynamical system — finite or infinite dimensional — with "enough" integrals of the motio all of whose Poisson brackets are zero, to solve th system in some "explicit" form. Now it has been t rather extraordinary experience in the field over the last 30 years, that many systems which are of great mathematical and physical interest, which may be Hamiltonian, and may not even be dynamical, c be solved "explicitly" using techniques that have direct link back to the method of solution for the KdV equation discovered by Gardner et al. kind of developments that I have in mind are for example

- the resolution of the classical Schottky problem in algebraic geometry in terms of the solution of

- the introduction of quantum groups

- integrable statistical models — connection to Jones polyn

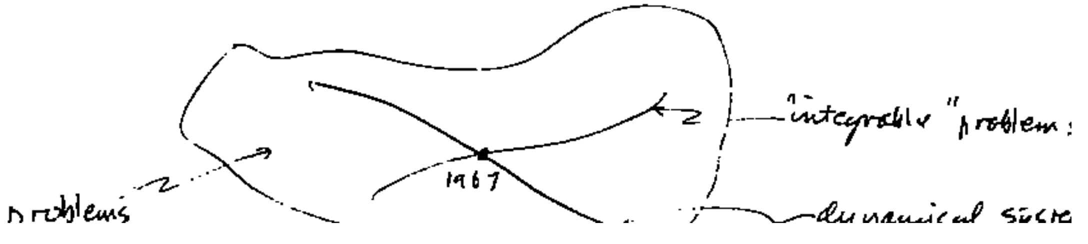
- 2D quantum gravity and the work of Witten and Kon
on the KdV hierarchy

- nonlinear special function theory (Painleve theory)
- conformal field Th^y — work of Krichever, Dubrovin
- * random matrix theory

- combinatorial problems of Robinson - Schensted - Knuth

The thrust of these lectures is to illustrate how the integrable methods interact with one of these top — random matrix theory.

Allow me a very schematic moment. Consider the following picture



In 1967, these two "manifolds" had a "transvers

intersection". The initial thought was that to dev

the ideas of Gardner et al. one should move alo

the "pde" manifold. But this turned out to be

too limiting: we now know that in order to se

the full development of the method one should move

in the "transverse" direction. And this is the direction we will move in the next 2+ lectures.

RMT

Now what is random matrix theory? RM

was first studied in the 1930's in the context of

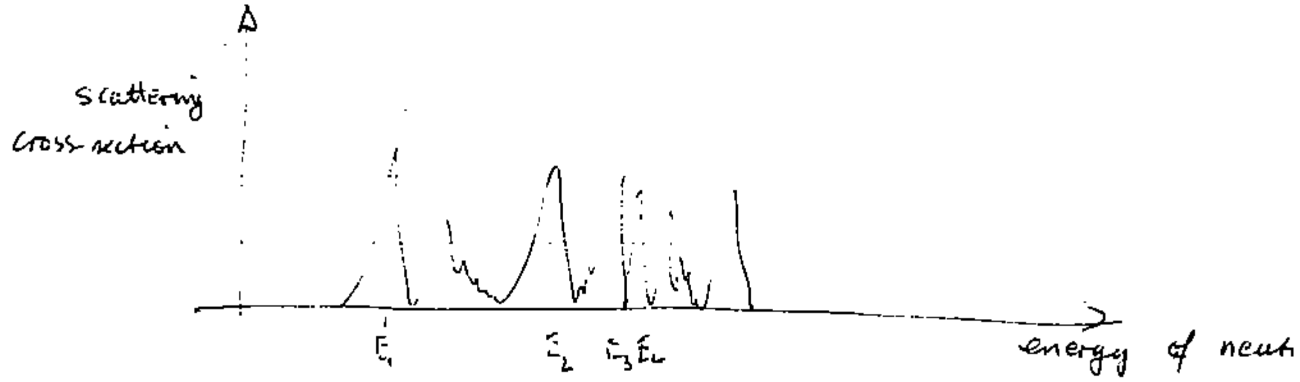
mathematical statistics by Hsu, Wishart and other

but it was Wigner in the early 1950's who

introduced RMT to mathematical physics. Wigner

was concerned with the scattering of neutrons off lar nuclei

Schematically one sees the following:



The energies E_i for which one obtains a large scatte cross section are called resonances, and for a heavy nucleus like U^{238}, say, there can be hundreds / t of them. In theory, one could write down non Schrödinger equation for the neutrons + nucleus a try to solve it numerically to compute these E_i But in Wigner's time and still in our time .

also into the foreseeable future, this was not a reali approach, and so people began to think that it wa more appropriate to give the resonances a statistica *meaning*. But what should the statistical model At this point Wigner put forward the *remarkable* *hypothesis* that the (high) resonances E_i *behave* the eigenvalues of a (large) matrix. It is l to overemphasize what a radical and revolutionary tho this was: we all recall that when we first wer learning some physics, we understood that the *detail* of the model were paramount: if you changed the force law in the equations of motion, the *behavior* of the system would change. But now all of that was out the window. The precise mechanism was no longer important. All that Wigner (, and later Dyson,) required was tha

- the matrices be <u>Hermitean</u> (or real symmetric) (so that the eigenvalues we real)

- that the ensemble behaves "naturally" under certain physical symmetry groups ($\overset{N\times N\ \text{Hermitean}}{\text{GUE}}$, $\overset{N\times N\ \text{real symmetric}}{\text{GOE}}$, $\overset{2N\times 2N\ \text{Hermitean, s}\ H = JH^T}{\text{GSE}}$ ensemb — more later

During the late 50's, 60's and early 70's, various research (particularly, Bohigas et al) began to <u>test Wigner's hypothesis</u> against real experimen data in a variety of physical situations and the results were pretty impressive. A classic reference for RMT is

- <u>Random</u> matrices — M. L. Mehta, 2nd Ed

and I invite you to look at p17 for a <u>comp</u> of <u>1726</u> nuclear level spacings against predictions of the GOE ensemble.

procedure: first we must rescale the resonances a

the eigenvalues of the random matrices so that

number of resences / unit interval

=

expected number of eigenvalues / unit interval

Then we compare the statistics.

More concretely, we typically have a situation w

we are interested in resonances in an interval $(E, E$

where $\delta E << E$ but δE contains many reson

We then rescale the resonances in δE so that t

average density is one. We then do a similar

scaling for the eigenvalues in some ensemble, a

only at that point, after we have adjusted

our "microscopes" do we compare statistics. A scient

approaches such problems with two instruments in

— a microscope and a list of ensembles

- the dials on the microscope are adjusted to account for the macros
  of the situation, and vary from system to system,

- but once the "slide" is in focus, one sees universal behavior described by one of the entries in the list of ensembles.

Having given me a schematic moment, I now ask you for a small philosophical moment.

Now it is always something of a mysterious process when begin to think of something that is quite deterministic in a statistical way. Nevertheless, i

is a process with which we are very familiar. W

take a die, for example, which surely obeys the laws of solid mechanics and aerodynamics, but we readily and intuitively understan

as a statistical object. Moreover we "know" what the stochastic model should be: all 6 sides have eq

probability. Looked at from this point of view, wha

Wigner had to do was to see the neutron + nucle for its stochasticity. In all such problems there is a some singular / asymptotic process involved and when we cross some "Bayesian" point, phase space opens up, and all bets are o We are standing, as it were, on the corner of B'u and 42nd Street and we are watching this little kid play 3 card Monte: if we are fast enough, we can follow all his moves but then "poof!" the card are out there, and all 3 cards are equally likely

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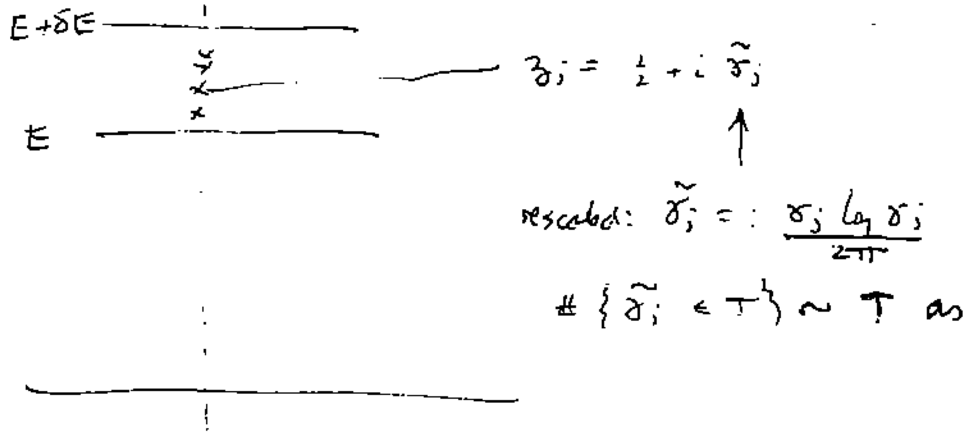
Now in the early 1970's a very remarkable thing happened. Montgomery, quite independently of the other goings on, began thinking that the zeros of the Riemann zeta function on the critical line $\mathrm{Re}\, z = \frac{1}{2}$, $z_j = \frac{1}{2}$ should also be viewed statistically. And so, assumi

and rescale the imaginary parts of the zeros

(·

$$\left(= \frac{1}{N} \# \{ \text{pairs } (i,j),\ 1 \le i \neq j \le N,\ \text{st } a \le \tilde{\gamma}_i - \tilde{\gamma}_j \le b \} \right)$$

for the (rescaled) zeros in an interval $(E, E+\delta E) \subset$ $E >> 1$,



$z_j = \frac{1}{2} + i\tilde{\gamma}_j$

rescaled: $\tilde{\gamma}_j =: \frac{\gamma_j \log \gamma_j}{2\pi}$

$\# \{ \tilde{\gamma}_j \le T \} \sim T$ as

and he found a limiting formula $R = R(a,b)$

$$\left(\int_a^b \left[1 - \left(\frac{\sin 2\pi u}{2\pi u} \right)^2 \right] du \right)$$

for the 2 pt. function as $E \to \infty$, $\# \tilde{\gamma}_j \in \delta E \to \infty$. W happened next is very well-known (I even asked Dyson to authenticate this version). Montgomery met Dyson at i at the Institute in Princeton, and when he told him about his calculations, Dyson immediately wrote down a formula and asked Montgomery, "Did you get this how he knew the answer, Dyson said "Well, *if* the zer of the zeta function *behaved* like the eigenvalues of a random *GUE matrix*, this would have to be the answer!" *Indeed* what Montgomery had obtained for R was *precisely* the 2-pt function for the eigenvalues of a (large) random (GUE) matrix.

At this point, the *cat* was out of the bag. People began asking whether their favorite list of #'s behaved like the eigenvalues of a random matrix. Through the 80's an *extraordinary* variety of systems were investigated from this point of view with *astounding* results: for example, if we take a region in the plane

and look at the eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots$ of the Dirichlet Laplacian in this region they they too (after the standard scaling) behave like the eigenvalu of a random GUE matrix (see Mehta p13)

In the late 1980's, Montgomery's work was t up numerically by Odlyzko, who then confirmed Montgomery's work to high accuracy & also investiga other statistics such as the nearest neighbor spacing again there was incredible agreement with random matrix theory. In recent years, the work of Montgom Odlyzko has been a wonderful springboard for Sarnak - Rudnick and then Sarnak - Katz, to prove all. kinds of GUE (& GSE) random matrix properties for the zeros of all kinds of automorphic L functions

Now up until very recently, the physical and mathematical phenomena which were investigated, concerne the eigenvalues of a random matrix in the bul of the spectrum.

But in the last 2 yrs or so a very interesting class of problems have started to appear in combina and also in statistical particle models, which concern the eigenvalues at the top of the spectrum. For example, consider the following version of solitaire called "patience sorting". Suppose we have N cards numbered linearly 1, 2, .., N for convenience. Shuffle the cards

$$1 \quad 2 \quad 3 \quad - \quad - \quad N$$

$$\pi \downarrow$$

Now take the top card $\pi(1)$ . . . .

Eventually you end up with $P_N(\pi)$ piles. Question

Putting uniform distribution on the set of ~~permut~~ shuffles $\{\pi\} = S_N$, how does $P_N(\pi)$ behave as $N \to \infty$? Or said differently, how big a table do I need typically to play the game with N cards?

~~To state ~~the~~ answer to ~~this~~ question we need introduce some notation~~

(which we will discuss in the 4th lecture)

The theorem is the following (Baik, Johansson, P.D.)

As $N \to \infty$, P_N, suitably centred and scaled behaves statistically like the largest eigenvalue of a random GUE matrix

More precisely suppose $\lambda^{(N)}(M) \geq \lambda^{(N)}(M) \geq \cdots \geq \lambda^{(N)}(M)$

are the eigenvalues of a (large) N × N GUE matr

then

$$\lim_{N\to\infty} \text{Prob}\left(\frac{P_N - 2\sqrt{N}}{N^{1/6}} \le t\right) = \lim_{N\to\infty} \text{Prob}\, \frac{(\lambda_1^{(N)}(M)) - \sqrt{\ }}{2^{-1/2}\, N^{-1/}}$$

$$\equiv F(t)$$

F(t) is called the Tracy - Widom distribution after it discovered and can be explicitly expressed in terms of a solution of the Painlevé II equation.

There are many equivalent formulations of the patience sorting problem – longest increasing subsequence of a random permutation, the height of a nucleat droplet in a supersaturated medium in the so-called Polynuclear Growth (PNG) model, the number of boxes in the first row of a Young diagram ~~w~~ under Plancherel measu

out that the # of boxes in the 2nd row behaves statistically like the 2nd largest eigenvalue $\lambda_2^{(N)}$ (M of a random GUE matrix, and so on.

Now what are these three basic ~~[illegible]~~ distributions mentioned above that (were) singled out by Wigner and Dyson (on the basis of the behavior of the system under time reversal and change of ~~[illegible]~~ reference frame)?

* i) Gaussian Unitary ensemble (GUE) consisting of

(a) N×N Hermitean matrices $M = (M_{ij})$

(b) with probability distribution

$$P(M)dM = \underset{\uparrow \atop \text{normalization density}}{P(M)} \prod_{1}^{N} dM_{ii} \prod_{i<j} d\mathrm{Re}M_{ij} \prod_{i<j} d\mathrm{Im}\,M$$

which is invariant under ~~automorph~~ $\underset{\overset{\equiv}{\mathcal{U}}}{\text{unitary}}$ conjugation

$$M \to U^{-1} M U = M', \quad U \text{ unitary}$$

$$\text{i.e. } P(M')\,dM' = P(M)\,dM$$

(c) $(M_{ii}, \mathrm{Re}\,M_{ij}, \mathrm{Im}\,M_{ij})$ are independent so

$$P(M) = \prod^{N} f_i^{(0)}(M_{ii}) \prod f^{(0)}(\overset{\mathrm{Re}}{M_{ij}}) \prod f^{(1)}(\overset{\mathrm{Im}}{M_{ij}})$$

2) Gaussian Orthogonal ensemble (GOE)

(a) $N \times N$ real symmetric

$\vdots$

3) Gaussian Symplectic ensemble (GSE)

(a) $2N \times 2N$ Hemitean self dual $M = M^*$

$M = J M^T J^{-1}$

$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

(b) . . .

invariant under $M \mapsto U^{-1} M U$, U unit
Symplecti

$U^T J U = J$

(c) . . .

The analysis of GUE is the simplest and I am going t
restrict myself to this case ~~in~~ throughout these lectures ...

So focusing on GUE, the first Theorem in the busine
is that if $P(M)$ satisfies (a) (b) (c) then necessarily

$$P(M) = \text{const } e^{-\operatorname{tr}(\alpha M^2 + \beta M + \gamma)}$$

where $\alpha > 0$, $\beta, \gamma \in \mathbb{R}$. Centering and rescaling we have the

$\underset{\uparrow \atop \text{gaussian}}{G}$UE distribution

(2) $$P(M) = \frac{1}{Z_N} e^{-\operatorname{tr} M^2} dM$$

Now here comes the problem: whereas conditions (a) & (b) are physical, (c) is just a device and has no physical basis.

If we just assume (a) and (b) we find that

$$\underline{(3)} \quad P(M)\,dM = \frac{1}{Z_N'} e^{-\operatorname{tr} V(M)}\, dM$$

for some real valued function V, $V(x) \to \infty$ as $|x| \to$

Ensembles of matrices of type (3) (are called Unitary Ensembles). Now physicists turn this all on its head and say "as $\nexists$ there is no physical way to distinguish between different choices of V, then whatever answers to physical questions that we compute, the answer must be independent of V." This is a rough form of what is meant by universality in random matrix theory. In the coming lectures we will focus on a particular basic statisti

The probability $\tilde{P}^{(V)}(a,b)$ that a (large) matrix has no eigen in an interval (a,b), and we will show that $P^{(V)}(a,b)$ indeed has a universal form independent of V. And in proving this, we will employ techniques that have arisen in integrable theory over the years and can be traced back in some form to the ~~original~~ original method of solution for the KdV equation discovered in 1967 Gardner, Greene, Kruskal and Miura.

To summarize, what has emerged, somewhat mysteriously and to everyone's surprise is a very powerful heuristic: all kinds of things physical and mathematical, behave, in some limit, like random matrices. Next ti you have a system that you want to ~~approximate~~ model in such a way wi a large matrix, go for it. You have a good chance of be