Qualifying Examination in Linear Algebra
August 2014

Instructions:
Choose SEVEN of the eight problems; only write solutions for these seven.
Please write neatly.
Please be sure to communicate your reasoning clearly.
Good luck!

Notation:
\( \mathbb{R} = \) the field of real numbers.
\( M^T \) is the transpose of a matrix \( M \).
\( \langle x, y \rangle \) denotes an inner product of two vectors.

Scoring:
Each of the seven problems will count 10 points.
Examination Problems

1. Let $A_{12}$ be a real $12 \times 12$ matrix having an eigenvalue $\lambda = 0$ with a geometric multiplicity of 5.
   
   (a) Determine the rank of $A$.
   
   (b) Suppose that the minimal polynomial of $A_{12}$ is $p_A(\lambda) = \lambda^3(\lambda + 4)^2$. Is $A_{12}$ diagonalizable? Justify your answer.

2. Let $f(x) = ((x - 1)^2 + 1)(x + 2)^2 + x$ and consider the matrix
   
   $$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & -1 \\ 1 & 0 & 1 \end{pmatrix}. $$

   Determine the eigenvalues of $f(A)$.


4. Let $A$ be a real $n \times n$ matrix with an eigenvalue $\lambda$ having algebraic multiplicity $n$. Show that
   
   $$e^{At} = e^{\lambda t} \left( I + (A - \lambda I)t + \cdots + \frac{(A - \lambda I)^{n-1}}{(n-1)!} t^{n-1} \right).$$

5. Describe the set of all eigenvectors for the matrix $A = \begin{pmatrix} -3 & 0 & -15 \\ 10 & 2 & 29 \\ 0 & 0 & 2 \end{pmatrix}$.

6. What sizes are the Jordan blocks of the matrix $A$ in problem 5? Justify your response.

7. Let $P$ and $Q$ be real idempotent square matrices of size $n$ such that $PQ = QP = 0$ and $P + Q$ is the identity. Show that $\mathbb{R}^n$ is the direct sum of the images of $P$ and $Q$.

8. Let $A$ be a matrix of size $n \times m$ and $B$ a matrix of size $m \times k$. Assume that $AB = 0$. Prove that the ranks of $A$ and $B$ sum to at most $m$. 