Duke University
Department of Mathematics

Qualifying Examination in Linear Algebra

Fall examination

Saturday, August 21, 2010

Instructions:

Choose **SEVEN** of the eight problems; only write solutions for these seven.

Please write neatly.

Please be sure to communicate your reasoning clearly.

Good luck!

Notation:

$\mathbb{R}$ = the field of real numbers.

Scoring:

Each of the seven problems will count 10 points.

Examination Problems

1. Let $\mathbf{A}$ be a 5×5 real matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = \pi$, $\lambda_4 = \pi^2$, and $\lambda_5 = \ln \pi$.

   (a) Is A invertible?

   (b) What are trace(A) and $\det(e^A)$?

   (c) Is A diagonalizable?

   (d) What are the eigenvalues of $\mathbf{A}^{2010}$?

   (e) Does the nullspace of $\mathbf{A}$ equal the orthogonal complement of the row space of $\mathbf{A}$?

2. Find a basis for the nullspace of

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

3. Suppose that V is an inner product space and W is a subspace of V. The orthogonal complement of W is defined by:

$$W^\perp = \{v \in V : \forall w \in W \ , \ \langle v, w \rangle = 0\}.$$

   (a) Show that $W^\perp$ is a subspace of V.

   (b) Show that $V = W \oplus W^\perp$.

   (c) Show that $W \subset (W^\perp)^\perp$.

   (d) Show that if V has finite dimension, then $W = (W^\perp)^\perp$.

4. Let W be the subspace of $\mathbb{R}^5$ spanned by the following vectors:

$$\begin{aligned} \mathbf{w}_1 &= (2, 2, -1, 0, 1) \\ \mathbf{w}_2 &= (-1, -1, 2, -3, 1) \\ \mathbf{w}_3 &= (1, 1, -2, 0, -1) \\ \mathbf{w}_4 &= (0, 0, 1, 1, 1). \end{aligned}$$

   Find a basis for $W^\perp$.

5. (a) Find the minimal polynomials of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

   (b) Find the characteristic polynomials of A and B.

   (c) Explain why or why not, A and B are similar.

6. Let $\mathbf{J}$ be a Jordan form of a matrix $\mathbf{A}$. Suppose that the algebraic multiplicity of the eigenvalues λ_0 is 3 and the geometric multiplicity of λ_0 is 2.

   (a) How many times does λ_0 occur on the diagonal of $\mathbf{J}$?

   (b) How many Jordan blocks contain λ_0?

   (c) Find a Jordan form for the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

7. Suppose that V is a vector space with subspaces $W_1, \ldots W_r$ so that $V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$. For $v = w_1 + w_2 + \cdots + w_r \in V$ with $w_i \in W_i$, define $E_i v = w_i$.

   (a) Prove that for all $1 \leq i \leq r$, $E_i E_i = E_i$.

   (b) Prove that for all $1 \leq i, j \leq r$ with $i \neq j$, $E_i E_j = 0$.

   (c) Prove that $E_1 + \ldots + E_r = I$.

8. Suppose that V is a real finite-dimensional inner product space and $P : V \to V$ is a linear transformation. Show that the following are equivalent:

   (a) There exists $T = T^\top$ so that $P = T^2$.

   (b) $P = P^\top$ and for all $v \in V$, the inner product $\langle Pv, v \rangle$ is non-negative.