Qualifying Examination in Linear Algebra

Fall examination
Saturday, August 21, 2010

Instructions:
Choose SEVEN of the eight problems; only write solutions for these seven.
Please write neatly.
Please be sure to communicate your reasoning clearly.
Good luck!

Notation:
\( \mathbb{R} \) = the field of real numbers.

Scoring:
Each of the seven problems will count 10 points.
Examination Problems

1. Let $A$ be a $5 \times 5$ real matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = \pi$, $\lambda_4 = \pi^2$, and $\lambda_5 = \ln \pi$.
   (a) Is $A$ invertible?
   (b) What are $\text{trace}(A)$ and $\det(e^A)$?
   (c) Is $A$ diagonalizable?
   (d) What are the eigenvalues of $A^{2010}$?
   (e) Does the nullspace of $A$ equal the orthogonal complement of the row space of $A$?

2. Find a basis for the nullspace of

\[
A = \begin{pmatrix}
2 & 2 & -1 & 0 & 1 \\
-1 & -1 & 2 & -3 & 1 \\
1 & 1 & -2 & 0 & -1 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]

3. Suppose that $V$ is an inner product space and $W$ is a subspace of $V$. The orthogonal complement of $W$ is defined by:

\[
W^\perp = \{ v \in V : \forall w \in W, \langle v, w \rangle = 0 \}.
\]
   (a) Show that $W^\perp$ is a subspace of $V$.
   (b) Show that $V = W \oplus W^\perp$.
   (c) Show that $W \subset (W^\perp)^\perp$.
   (d) Show that if $V$ has finite dimension, then $W = (W^\perp)^\perp$.

4. Let $W$ be the subspace of $\mathbb{R}^5$ spanned by the following vectors:

\[
\begin{align*}
w_1 &= (2, 2, -1, 0, 1) \\
w_2 &= (-1, -1, 2, -3, 1) \\
w_3 &= (1, 1, -2, 0, -1) \\
w_4 &= (0, 0, 1, 1, 1).
\end{align*}
\]

Find a basis for $W^\perp$.  

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5. (a) Find the minimal polynomials of \( A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \).

(b) Find the characteristic polynomials of \( A \) and \( B \).

(c) Explain why or why not, \( A \) and \( B \) are similar.

6. Let \( J \) be a Jordan form of a matrix \( A \). Suppose that the algebraic multiplicity of the eigenvalues \( \lambda_0 \) is 3 and the geometric multiplicity of \( \lambda_0 \) is 2.

(a) How many times does \( \lambda_0 \) occur on the diagonal of \( J \)?

(b) How many Jordan blocks contain \( \lambda_0 \)?

(c) Find a Jordan form for the matrix \( A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \).

7. Suppose that \( V \) is a vector space with subspaces \( W_1, \ldots, W_r \) so that \( V = W_1 \oplus W_2 \oplus \cdots \oplus W_r \). For \( v = w_1 + w_2 + \cdots + w_r \in V \) with \( w_i \in W_i \), define \( E_i v = w_i \).

(a) Prove that for all \( 1 \leq i \leq r \), \( E_i E_i = E_i \).

(b) Prove that for all \( 1 \leq i, j \leq r \) with \( i \neq j \), \( E_i E_j = 0 \).

(c) Prove that \( E_1 + \cdots + E_r = I \).

8. Suppose that \( V \) is a real finite-dimensional inner product space and \( P : V \to V \) is a linear transformation. Show that the following are equivalent:

(a) There exists \( T = T^\top \) so that \( P = T^2 \).

(b) \( P = P^\top \) and for all \( v \in V \), the inner product \( \langle Pv, v \rangle \) is non-negative.