Math 32L
Final Exam

May 6, 2000

NAME:____

INSTRUCTOR:________________________________ SECTION: ______________

Instructions:

This exam has nine pages in addition to this cover sheet. For grading purposes the point value of the entire exam is 200 points. The values of the individual questions are indicated beside the questions. We suggest that you spend not more than 17 minutes on each page. If you follow this strategy for time allocation, then you will have almost 30 minutes at the end to use for checking, refinement, and correction.

Some of the exam questions include carefully worded descriptions or directions. In order to complete the problems correctly, you may need to read these instructions carefully before attempting the problems. You may use your calculator and the one–page set of notes that you prepared for this exam. ***You are expected to show your work****. In particular if you use the calculator to assist in answering a question, be sure to reproduce (by hand) any relevant graphs and explain any relevant mathematics.*

When you turn in your exam, <u>be sure that each sheet has your name and section number on it</u> and that you have signed the honor code below.

Honor Code:

I have neither given nor received aid in the completion of this examination.

Signature: ____

Please do not write below this line.

Page 1 __________	Page 4 __________	Page 7 __________
Page 2 __________	Page 5 __________	Page 8 __________
Page 3 __________	Page 6 __________	Page 9 __________

TOTAL: [ ]

1. (20 points) Experience has shown a West Texas oil company that, if they drill a well in a particular area, the probability of striking oil is 0.06. If oil is found, a typical well is worth 10 million dollars minus the cost of drilling.

(a) If three wells are drilled, what is the probability that exactly one of the wells produces oil? (The probability of striking oil in one location is independent of striking oil at another location.)

(b) If three wells are drilled, what is the probability that at least one of the wells produces oil?

(c) If the cost for drilling a well is $300,000, what is the expected profit for this company if they drill three wells?

(d) The oil company has discovered that the probability of striking oil goes down by $\frac{1}{3}$ for every 100 feet drilled. Thus, if p is the probability of striking oil in the first 100 feet of drilling, then $\frac{1}{3}p$ is the probability of striking oil in the next 100 feet, and so on. Using the facts that the probability of striking oil in a particular well is 0.06 and assuming for the purpose of this model that we can drill a well of infinite depth, find the probability of striking oil in the first 100 feet.

2. (20 points) Some engineers working for Emtel designed a new memory chip. They hypothesized that the failure time for this chip could be described by the density function

$$p(t) = \begin{cases} \frac{c}{\sqrt{(t+4)}} & \text{for } 0 \le t \le 12 \\ 0 & \text{for all other } t \end{cases}$$, where c is a constant and t is measured in years.

(a) Show that $c = \frac{1}{4}$.

(b) Write a formula for the cumulative distribution function for the failure times.

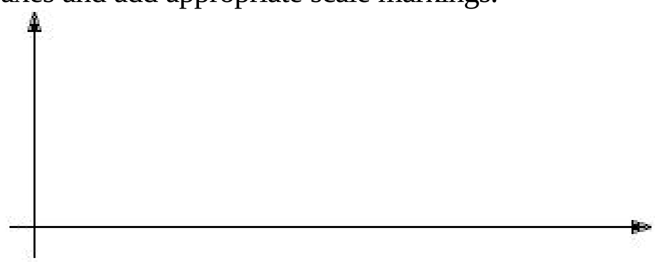
(c) Find the anticipated median failure time for the chips.

(d) Find the anticipated mean failure time.

3. (10 points) A new statistician just arrived at Emtel, and doubting the assumptions made in the previous problem, she decides to construct a cumulative distribution function directly from reliable data. She gathered data on the performance of the chips in 25 randomly chosen computers, which had been tracked over a 13-year period. A record of the number of chips which failed in a given year is shown in the following table:

Year	1	2	3	4	5	6	7	8	9	10	11	12	13
Number of chip failures in the given year	4	3	3	2	2	2	2	2	1	1	1	0	0

As the new statistician did, sketch the graph of the cumulative distribution function <u>using the data</u> in the table above. Be sure to label the axes and add appropriate scale markings.

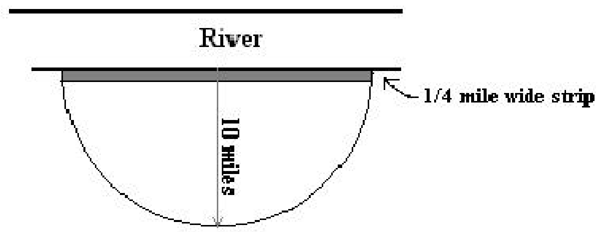


4. (14 points) A storage tank is constructed in the shape of a large rectangular box with dimensions 6 feet by 4 feet by 8 feet. The tank is to hold a fluid weighing 55 lbs/ft^3. The tank is to be buried so that its top surface is 10 feet below the ground. As the tank is readied for burial, an argument erupts among the workers as to whether the work required to pump the fluid out of the tank would be different if the 4 ft by 8 ft surface were at the top, or the 6 ft by 8 ft surface were at the top. Calculate the work required to pump the fluid out of the tank in both cases, and settle the argument.

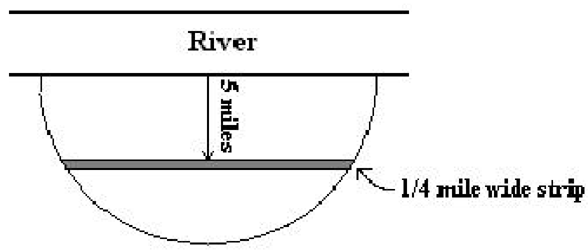
5. (12 points) A nature park in the shape of a semicircle of radius 10 is situated such that its diameter runs along a straight river. Using mosquito traps set up around the park, entomologists have determined that the mosquito population density, expressed in units of 10,000 mosquitoes per square mile, can be modeled by the function

$$f(x) = 500e^{-.1x}\ ,$$ where x is the distance in miles from the river (measured in a direction <u>perpendicular</u> to the river).

(a) Estimate the number of mosquitoes on the quarter mile wide strip of park land adjacent to the river as shown in the picture at the right.



(b) Estimate the number of mosquitoes on the quarter mile wide strip of park land located five miles back from the river, as shown in the picture at the right.



(c) In the space to the right set up an integral that represents the total number of mosquitoes in this park. **Do not evaluate the integral.**

6. (10 points) When wildlife officials closed a lake to fishing, the population of trout in the lake grew logistically (i.e., restricted growth) with the constant of proportionality 0.125 and a carrying capacity of 100,000. (Carrying capacity is defined to be the lake's maximum sustainable trout population.) Recently the officials decided to allow fishing on the lake again, and they estimated that the trout will be harvested at a continuous annual rate of 6%.

(a) Construct a differential equation which describes the trout population after fishing resumed.

(b) What is the long term effect fishing will have on the trout? Be specific and justify your answer.

7. (15 points) For each series below, determine if it converges or diverges. If the series converges, find its exact value if you can; if you cannot find its exact value, approximate it by using the fourth partial sum and find a reasonable error bound. Show clearly the justification for your answers.

(a) $\sum_{k=0}^{\infty} \left(\frac{8}{7}\right)^k$

(b) $\sum_{k=2}^{\infty} \frac{1}{k(lnk)^2}$

(c) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(\pi)^{2k}}{(2k)!}$

8. (10 points) Consider the following series: $\sum_{k=1}^{\infty} \left(7^{\frac{1}{k}} - 7^{\frac{1}{k+2}}\right)$

(a) Find the 4^{th} partial sum of the above series. Simplify your answer.

(b) Find a formula for the n^{th} partial sum of this series.

(c) Does this series converge? If so, find its exact sum; if not, explain why not.

9. (16 points) (a) Find the Taylor series centered at 0 for $y = e^{-x}$. Show at least **seven non-zero** terms.

(b) Give the first **five non-zero** terms of the Taylor series centered at 0 for $f(x) = \frac{1-e^{-x}}{x}$.

(c) Use your answer from part (b) to find a series that converges to the value of $\int_0^1 \frac{1-e^{-x}}{x}\,dx$. Express your answer using Σ-notation.

(d) How many terms of the series found in part (c) would be required to approximate the value of the integral with an error less than 0.0003? You must justify your answer.

10. (10 points) Find all values of x for which the series $\sum_{k=1}^{\infty}(-1)^k \frac{x^{2k}}{4^k\sqrt{k}}$ will converge.

11. (20 points) Analysts refer to companies whose stock price appears to fluctuate periodically as *cyclicals*. In the table below the value of a single share of stock for one such company has been recorded. Assume that this *cyclical* stock has period 2π years.

t (years)	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
price/share	\$23.00	\$42.00	\$56.00	\$30.00	\$18.00	\$20.00	\$23.00

(a) We will construct a second-order Fourier polynomial which models the given data. The values of b_1 and a_2 have already been estimated for you. You should compute estimates of a_0, a_1, and b_2, and write your estimates in the appropriate blanks below. Also, indicate which method of approximation you choose to use, and show clearly any expressions that you compute. Finally, you should write out the entire second-order Fourier polynomial.

Approximation method: ____________________

$a_0 =$ ________

$a_1 =$ ________

$b_1 =$ <u>17.32</u>

$a_2 =$ <u>−5</u>

$b_2 =$ ________

Fourier polynomial:

$f(t) =$ ____

(b) The *volatility* of a stock is defined to be the maximum of the absolute value of the derivative of its stock price function. Make a sketch below of the graph of the Fourier polynomial you created in part (a). *Estimate* visually a location on your graph where we can measure the volatility of this stock. [You do not need to make the estimate.]

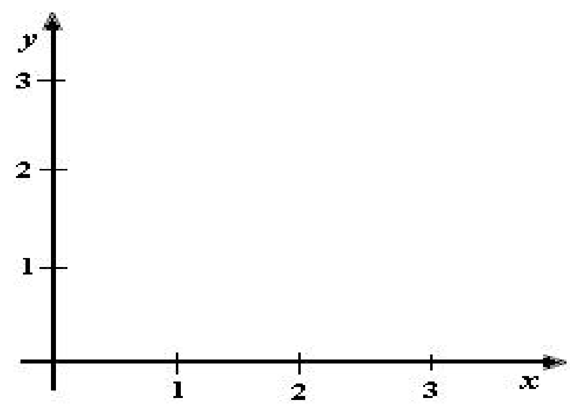
12. (19 points) During World War One F.W. Lanchester constructed mathematical models to describe the course of battles. The model Lanchester proposed for a conventional army x battling a guerilla army y, where both sides receive reinforcements at a constant rate, is

$$\frac{dx}{dt} = -y + 1 \qquad \frac{dy}{dt} = -x - xy + 2,$$

where $x(t)$ and $y(t)$ denote the number of combatants (in units of 10,000) of the x and y forces at time t, where t is measured in days from the start of the battle.

(a) Find an expression for $\frac{dy}{dx}$.

(b) Find the nullclines and equilibrium points for this system of differential equations and sketch the phase plane for this system.



(c) Describe the course of a battle modeled by the above equations, given that $x(0) = 30{,}000$ troops and $y(0) = 20{,}000$ troops.

13. (8 points) Circle every function below which is a solution of the differential equation

$$\frac{d^2y}{dt^2} = 9y \qquad \text{(Note that the constant 9 is positive.)}$$

$y = 6sin(3t)$	$y = 3cos(9t)$	$y = 2e^{3t}$	$y = 3e^{9t}$
$y = 7e^{-3t}$	$y = 2sin(9t)$	$y = 3e^{-9t}$	$y = 2e^{3t} + e^{-3t}$

14. (8 points) The three population models that you studied this semester are the following:

I. *Exponential growth* **II.** *Logistic (restricted) growth* **III.** *Coalition (doomsday) growth*

Beside each of the following descriptions, write the Roman numeral(s) corresponding to the model(s) which match the description. More that one model may match a given description.

(a) ______ The relative growth rate, $\frac{dP}{dt}/P$, is a linear decreasing function of the population.

(b)______ The graph of the solution contains an asymptote.

(c)______ The model assumes the population has unlimited resources.

(d)______ The relative growth rate, $\frac{dP}{dt}/P$, remains constant.

15. (4 points) Consider the periodic function, having one period and defined by $y = 4 - t^2$ on the interval [-2,2]. The general form of the n^{th} harmonic of the Fourier series representation of this function is . . . [choose one answer]

A. $a_n cos(2\pi n t) + b_n sin(2\pi n t)$

B. $a_n cos(n\pi t) + b_n sin(n\pi t)$

C. $a_n cos(\frac{n\pi}{2} t) + b_n sin(\frac{n\pi}{2} t)$

D. $a_n cos(4\pi n t) + b_n sin(4\pi n t)$

19. (4 points) The third **non-zero** term of the Taylor series for $y = sin(x)$ centered about $x = \frac{\pi}{4}$ is. . . [choose one answer]

A. $\dfrac{\left(x - \frac{\pi}{4}\right)^5}{5!}$

B. $-\dfrac{\left(x - \frac{\pi}{4}\right)^5}{5!}$

C. $\dfrac{\sqrt{2}}{4}\left(x - \dfrac{\pi}{4}\right)^2$

D. $-\dfrac{\sqrt{2}}{4}\left(x - \dfrac{\pi}{4}\right)^2$

E. $\dfrac{\sqrt{2}}{2}\left(x - \dfrac{\pi}{4}\right)^5$