Series Solutions of Initial Value Problems

Purpose

You will learn how Taylor series can be used to solve some initial value problems.

Preview

We begin this method of solving initial value problems by assuming that the solution can be written as a Taylor series expanded about 0. We substitute a "generic" series into the differential equation and then determine what its coefficients must be. Once we know these coefficients, we have a solution!

Part 1

Consider the familiar initial value problem, $\frac{dy}{dt} = y$, with $y(0) = 2$. We start with this particular problem, because we already know the solution and thus we can verify the result of our work.

(a) Find the solution to this problem using the technique we learned last semester. Don't forget the initial condition.

Now, we will solve the same problem by using a Taylor series. First, we assume that the solution, y, can be written as a Taylor series of the following form:

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots + a_n t^n + \cdots$$

(b) Use the initial condition to determine a_0.

(c) Note that we can differentiate the series expression for y above to produce a series expression for $\frac{dy}{dt}$. Do that now, and then substitute both the series for y and the series for $\frac{dy}{dt}$ into the original differential equation. You now have an equation with a series on each side of the equation.

The only way the series on the right-hand side of your equation can equal the series on the left-hand side is if the coefficients of terms with corresponding powers of t are equal; i.e., the coefficient in front of the t on the left must equal the coefficient in front of the t on the right, and so on.

(d) Set the constant term on the right equal to the constant term on the left, and solve for a_1.

Set the coefficient of t on the right equal to the coefficient of t on the left, and solve for a_2.

Set the coefficient of t^2 on the right equal to the coefficient of t^2 on the left, and solve for a_3.

Continue this process to find a_4, a_5, a_6, and a_n. [Hint: to find a pattern, do not simplify your fractions; for example, write a_4 as $\frac{2}{2\cdot 3\cdot 4}$, not as $\frac{1}{12}$.]

(e) Now that you have found the coefficients, write out the series for y.

(f) If you factor out a 2 from your series for y, you should recognize the result as the Taylor series of a well-known function. Compare your result to the answer you computed in part (a).

(g) If we had started out with a Taylor series expanded about a point other than 0, would we still have been able to find the solution to this initial value problem? If not, explain the difficulty.

Part II

Consider the initial value problem $\frac{dy}{dt} = t + y$, with $y(0) = 1$. Unlike the previous initial value problem, this one cannot be solved with the techniques learned last semester. (Do you see why?) Once again assume that the solution can be written as a Taylor series, and follow the steps outlined in Part I. Be careful in setting the coefficients of the t terms equal to each other, because the right side has an extra t. Show clearly all of your work.

Once you have the series representation of the solution, y, you may find it difficult to recognize it as a familiar function. Indeed, we can**not** in general identify the series as a familiar function. Sometimes, however, we may be able to add terms or factor terms in such a way that we can recognize parts of the solution.

Example. Suppose that a series for a function, f, is given by the series

$$f(t) = 1 - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots .$$

This series looks somewhat like the Taylor series for $sin(t)$:

$$sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots$$

We can change f into the series for $sin(t)$ by subtracting 1 and adding t:

$$f(t) - 1 + t = \; t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \cdots = sin(t)$$

Thus,

$$f(t) = sin(t) - t + 1.$$

Now, use the ideas in the example above to write the series obtained in Part II in terms of a well-known function.

Part III

Up to this point of the course the differential equations with which we have worked have all be *first order*; i.e., they involved the first derivative of an unknown function. We shall now see that the series technique for solving differential equations can be used to solve initial value problems involving *second order* differential equations.

Consider the initial value problem

$$\frac{d^2y}{dt^2} = -y, \text{ with } y(0) = 1 \text{ and } y'(0) = 0.$$

Again assume that the solution y can be written as a Taylor series expanded about zero. The initial conditions will allow you to solve for a_0 and a_1. Use the method of equating coefficients, as we did earlier in the lab, to find the rest of the coefficients in the series that represents the solution.

Part IV

It was stated earlier in this lab that the solution to an initial value problem obtained from the series solution technique may not look familiar and will have to remain in the form of a series. This will be the case for the final example. Consider the following initial value problem:

$$\frac{d^2y}{dt^2} = -ty, \text{ with } y(0) = 0 \text{ and } y'(0) = 1.$$

(a) As in Part III, find a series solution to this initial value problem.

(b) Over the interval [0,3] graph the seventh degree polynomial approximation to your solution. (Make the range of your dependent variable [-0.5, 1.5].)

(c) Use your graph to estimate where the solution reaches its maximum on the interval [0,3]. If you use a tenth-degree polynomial approximation of your solution, will your answer change?

Report

Your report should include your responses to all the questions. Clearly show all of your calculations. Hand sketch the graph you created in Part IV. Do not include the following practice problems in your report.

Practice Problems

The following initial value problems are given for practice so that you can test whether you understand the ideas in this lab.

1. Solve the initial value problem:

$$xy'' + 2y' + xy = 0, \text{ with } y'(0) = 0 \text{ and } y(0) = 1.$$

$\left[\text{Solution: } y(x) = \frac{sin(x)}{x}\right]$

2. Find a series solution to the initial value problem $\frac{dy}{dt} = y - t$, with $y(0) = c$. Write all the coefficients in terms of $c - 1$, and then use the ideas from Part II to show that the function represented by the series is $y(t) = (c-1)e^t + t + 1$.

The graphs of the solutions to this differential equation take on three distinct shapes depending on the initial condition (in particular, whether $c > 1$, $c = 1$, or $c < 1$). The slope field below can help you to see the three cases.