Series #1: Limits of Partial Sums

Before beginning our study of infinite sums we first need to discuss briefly the convergence of sequences of real numbers. A sequence is just a list of numbers in a given order. Here are three examples:

$$
\begin{array}{l}
2,\;\; 4,\;\; 6,\;\; 8,\;\; 10,\;\; 12, \ldots \\
1,\;\; -1,\;\; 1,\;\; -1,\;\; 1,\;\; -1, \ldots \\
1,\;\; 1\tfrac{1}{2},\;\; 1\tfrac{3}{4},\;\; 1\tfrac{7}{8},\;\; 1\tfrac{15}{16}, \ldots
\end{array}
$$

Sometimes we give names to the terms in a sequence by letting c_1 denote the first term, c_2 denote the second term, and so forth. We represent the entire sequence by $\{c_n\}_{n=1}^{\infty}$, where n is the index which runs from 1 to ∞, indicating that the successive terms of the sequence are $c_1, c_2, c_3, \ldots$. In the case of the first sequence above, $c_n = 2n$.

Definition. We say that a sequence $\{c_n\}_{n=1}^{\infty}$ **converges** to a limit L if we can make c_n as close to L as we like by choosing n large. We shall sometimes use the notation

$$\lim_{n \to \infty} c_n \;=\; L$$

to indicate that the sequence $\{c_n\}_{n=1}^{\infty}$ converges to a limit L.

Let's see which of the sequences given above converge. The first sequence can't get close to any one number because each term is larger by 2 than the preceeding term. The second sequence can't get close to any one number because the terms oscillating between $+1$ and -1. The third sequence is

$$c_n \;=\; 2 \;-\; \left(\tfrac{1}{2}\right)^{n-1}$$

Since $(\frac{1}{2})^n$ gets smaller and smaller as n get larger, we see that c_n approaches the limit $L = 2$.

Example 1. Here are some more examples of convergent sequences.

(a) The sequence $c_n = \frac{1}{n}$ converges to 0.

(b) The sequence $c_n = 5 - 42^{-n} + \frac{6}{n}$ converges to 5.

(c) The sequence

$$c_n \;=\; \left(2 + \frac{1}{n}\right)\left(7 - \frac{1}{n^2}\right) \;-\; \left(5 - 2^{-n} + \frac{6}{n}\right)^2$$

converges to -11.

(d) The sequence

$$c_n = \frac{4+7n}{2+3n}$$

is a little more difficult. Divide numerator and denominator by n to obtain

$$c_n = \frac{\frac{4}{n}+7}{\frac{2}{n}+3}$$

Now we can see that as n gets large this sequence approaches $\frac{7}{3}$.

Now we are ready to turn our attention to series. A series is an infinite sum

$$a_1 + a_2 + a_3 + a_4 + a_5 + \quad \ldots \tag{1}$$

We have already seen examples of such sums in our study of probability. Since it is awkward to write out sums (whether finite or infinite), we shall use the summation notation

$$\sum_{j=n}^{m} a_j = a_n + a_{n+1} + \ldots + a_{m-1} + a_m.$$

Using this notation, the infinite sum in (1) would be written

$$\sum_{j=1}^{\infty} a_j.$$

There is a practical and theoretical problem with infinite sums: we cannot compute a sum by adding one term after another if there are infinitely many terms. We need another way to make sense of and to compute infinite sums. For this reason, we introduce partial sums and limits. We define the n^{th} **partial sum** of this series to be the sum of the first n terms

$$S_n = \sum_{j=1}^{n} a_j$$

of the series. Now we can say what we mean by the infinite sum (1).

Definition. We say that the infinite sum (1) **converges** if the sequence of partial sums $\{S_n\}$ converges to a finite limit S as n gets larger. In this case, we define

$$\sum_{j=1}^{\infty} a_j = S.$$

If the sequence of partial sums does not converge to a limit, we say that the series **diverges**.

This definition makes sense because it says that, as we add more and more terms to the sum, if we get closer and closer to a number S, then we define this number S to be the sum of the series.

Example 2. Let $r \neq 1$ and define $a_j = r^j$. This is the **geometric series** which we studied in the probability laboratory. The n^{th} partial sum is

$$S_n = 1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}$$

If $|r| < 1$, we can see that the sequence of partial sums S_n converges to $\frac{1}{1-r}$. Therefore, by definition, if $|r| < 1$, the series converges and

$$1 + r + r^2 + \ldots + r^{n-1} + \ldots = \frac{1}{1-r}$$

If $r = 1$, the n^{th} partial sum is

$$S_n = 1 + 1 + 1^2 + \ldots + 1^{n-1} = n$$

so the sequence of partial sums does not converge. Thus the geometric series does not converge if $r = 1$. If fact, the geometric series does not converge for any r satisfying $|r| \geq 1$ (see problem 2).

Example 3. Consider the series

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots. \tag{2}$$

Since

$$\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1},$$

we can compute the n^{th} partial sum explicity.

$$\begin{aligned} S_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Since $S_n = 1 - \frac{1}{n+1}$ we see that the sequence of partial sums converges to 1. Thus, the series (2) converges and

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} \;=\; 1.$$

Example 4. In Example 3 you may have been tempted to perform an infinite cancellation on the series $(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) \ldots$. The following example shows that such cancellation can give incorrect results. Consider the series

$$(5 \;-\; 5^{\frac{1}{2}}) \;+\; (5^{\frac{1}{2}} \;-\; 5^{\frac{1}{3}}) \;+\; (5^{\frac{1}{3}} \;-\; 5^{\frac{1}{4}}) \;+\; \ldots \;+\; (5^{\frac{1}{j}} \;-\; 5^{\frac{1}{j+1}}) \;+\; \ldots$$

The n^{th} partial sum is

$$S_n \;=\; 5 \;-\; 5^{\frac{1}{n+1}}$$

which converges to 4 as n gets larger and larger. The infinite cancellation gives 5, which is incorrect.

The series in Examples 2, 3, and 4 are very special. Unfortunately, for most series there is no way to calculate a simple formula for the n^{th} partial sum, S_n. If we can't calculate S_n, how are we ever going to be able to tell whether S_n converges? Good question! If we can somehow determine that a series converges, we can approximate the sum by a partial sum with high n. We shall see in the next lecture that there are some tests which one can apply to a series which guarantee that it converges if the test comes out positive. For the moment, we raise the question of how large the terms a_j can be if the sum is finite. We will see that "most of the terms " must be small in the following sense: if $\sum_{j=1}^{\infty} a_j$, then the *sequence* $a_1, a_2, a_3, \ldots$ must converge to zero. For any positive integer m, we can write

$$\begin{aligned} a_m \;&=\; S_m \;-\; S_{m-1} \\ &=\; (S \;-\; S_{m-1}) \;-\; (S \;-\; S_m). \end{aligned}$$

As m gets larger and larger, both terms on the right get smaller and smaller (since, for a convergent series, the partial sums converge to the sum). Thus a_m gets smaller and smaller as m gets larger and larger.

The n^{th} Term Test for Convergence. If a series $\sum_{j=1}^{\infty} a_j$ converges, then the sequence of terms, $\{a_j\}$, must converge to 0.

Example 5. This condition can sometimes be used to show that series do not converge. Consider the series

$$\sum_{j=1}^{\infty}(-1)^j$$

The j^{th} term is $a_j = (-1)^j$. Since the *sequence* $\{a_j\}$ does not converge to 0 (it oscillates between $+1$ and -1), the *series* can not converge.

The condition that $\{a_j\}$ converges to 0 is a *necessary* condition for a series to converge. That is, if the series converges, it *must* be true. However, it is *not* a sufficient condition for convergence. That is, just because $\{a_j\}$ converges to 0 does not prove that the series converges. Here is an example.

Example 6 (The harmonic series). Consider the series

$$\sum_{j=1}^{\infty}\frac{1}{j}.$$

Notice that $a_j = \frac{1}{j}$, so it is certainly true that the sequence $\{a_j\}$ converges to 0. However, we shall see that the series does *not* converge. We cleverly group the terms of the series in the following way:

$$\begin{aligned}
& 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}+\tfrac{1}{9}+\tfrac{1}{10}+\tfrac{1}{11}+\tfrac{1}{12}+\tfrac{1}{13}+\tfrac{1}{14}+\tfrac{1}{15}+\tfrac{1}{16}+\ldots \\
= \; & 1+\tfrac{1}{2}+\{\tfrac{1}{3}+\tfrac{1}{4}\}+\{\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\}+\{\tfrac{1}{9}+\tfrac{1}{10}+\tfrac{1}{11}+\tfrac{1}{12}+\tfrac{1}{13}+\tfrac{1}{14}+\tfrac{1}{15}+\tfrac{1}{16}\}+\ldots \\
\geq \; & 1+\tfrac{1}{2}+\quad\tfrac{1}{2}\quad+\quad\tfrac{1}{2}\quad+\quad\tfrac{1}{2}\quad+\ldots
\end{aligned}$$

This shows that the *sequence* partial sums increases without bound, so the *series* does *not* converge. Note that the series diverges even though the sequence of terms approaches zero.

A Note on Notation. We have been using the the letter j for the index in sums:

$$\sum_{j=n}^{m} a_j = a_n + a_{n+1} + \ldots + a_{m-1} + a_m.$$

If we used the letter k for the index, the sum

$$\sum_{k=n}^{m} a_k = a_n + a_{n+1} + \ldots + a_{m-1} + a_m$$

would still be the same. Thus, there is no difference between the series $\sum_{j=1}^{\infty}\frac{1}{j(j+1)}$ and the series $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ except for the name given to the index.

Problems.

(1) For each of the following sequences, say whether it converges and if it does say what the limit is.
   (a) $c_n = (3 - \frac{2}{n})$.
   (b) $c_n = (3 - \frac{2}{n})^2$.
   (c) $c_n = 3^n$.
   (d) $c_n = 3^n \frac{2}{n}$.
   (e) $c_n = 2 + (-1)^n$.
   (f) $c_n = 2 + \frac{(-1)^n}{n}$.
   (g) $c_n = (2 - \frac{(-1)^n}{n})(5 + \frac{17}{n}) \ - \ 4$.
   (h) $c_n = \frac{6-n}{2+3n}$.
   (i) $c_n = \frac{6-n+3n^2}{2+3n+15n^2}$.

(2) By looking at each of the cases, $r = 1$, $r = -1$, $r > 1$, $r < -1$, show that the geometric series does not converge if $|r| \geq 1$.

(3) Find the sum of the series

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k .$$

(4) Find the sum of the series

$$\sum_{k=1}^{\infty} \left(7^{\frac{1}{k}} - 7^{\frac{1}{k+1}}\right) .$$

(5) By using the idea in Example 3, show that

$$\sum_{k=2}^{\infty} \frac{1}{k^2 - 1} \ = \ \frac{3}{4}.$$

(6) We shall show later in the course that the series $\sum_{j=1}^{\infty} \frac{1}{j^2}$ converges. Use your calculator to approximate the sum accurate to two decimal places. Explain how you decide when you have the required accuracy.

(7) Compute S_{10}, S_{80}, and S_{700} for the series

$$4 \ - \ \frac{4}{3} \ + \ \frac{4}{5} \ - \ \frac{4}{7} \ + \dots$$

Do you think that this series has a sum? If so, what do you think it is?

(8) We showed in Example 6 that the harmonic series $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges. Use your hand calculator to compute S_{100} and S_{200}. Using the inequality from the divergence argument in Example 6, determine how large n would have to be so that S_n is greater than 10. What does this tell you about using a calculator to determine whether a series converges or diverges?

(9) Suppose that we change every other sign in the harmonic series to a minus sign obtaining the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

We shall prove later in the course that this new series converges. Use partial sums to approximate the sum. How many terms do you think you need to get 2 decimal place accuracy? Why do you think that the behavior of these partial sums is so different from the partial sums of the harmonic series?

(10) Use the necessary condition to prove that the following series do not converge:
  (a) $\sum_{j=1}^{\infty} 2^j$.
  (b) $\sum_{j=1}^{\infty} \sin\left(\frac{\pi}{4}j\right)$.
  (c) $\sum_{k=1}^{\infty}(1 + \frac{1}{k})$.

Answers to Selected Problems.

1. (g) converges to 6; (h) converges to $-\frac{1}{3}$; (i) converges to $\frac{1}{5}$.
3. 3.
7. $S_{10} = 3.041$, $S_{80} = 3.129$, $S_{700} = 3.140$.