Probability #3: Expectation and Options

In this lecture we introduce the concept of expectation (or average value) of a random variable and show through several examples how it can be used in gaming situations, economics, and the financial markets.

Consider an experiment in which we roll two dice many times and tally the sums of the numbers on the two dice. What would we "expect" the average of those rolls to be? To get the answer, we carry out a thought experiment. Let N be the number of times we roll the two dice. (We assume that N is very, very large.) Remembering the probability mass density computed in Example 3 of the last section, we expect that the sum 2 would appear about $\frac{1}{36}$ of the time. That is, we expect to roll 2's approximately $\frac{1}{36}N$ times. Since the sum 3 would appear about $\frac{2}{36}$ of the time, we expect to roll 3's about $\frac{2}{36}N$ times. Continuing this line of thought, we expect 4's about $\frac{3}{36}N$ times, 5's about $\frac{4}{36}N$ times, 6's about $\frac{5}{36}N$ times, etc. Now, we compute the average of all expected rolls using the usual definition of average (i.e. take the sum of all the rolls and divide by N):

$$\begin{aligned}\text{(Expected) Average} &= \frac{\frac{1}{36}N(2) + \frac{2}{36}N(3) + \frac{3}{36}N(4) + \ldots + \frac{1}{36}N(12)}{N} \\ &= \frac{1}{36}(2) + \frac{2}{36}(3) + \frac{3}{36}(4) + \ldots + \frac{1}{36}(12)\end{aligned}$$

A remarkable thing happened: the N canceled out of the equation! Thus, the average value of the rolls we expect to get is

$$\frac{1}{36}(2) + \frac{2}{36}(3) + \frac{3}{36}(4) + \ldots + \frac{1}{36}(12) \ = \ 7$$

We call 7 the "expected value" of rolling two dice; i.e., if we roll two dice a very large number of times, we would expect the average of the rolls to be 7. (Of course, you may roll two dice 10 billion times, compute the average, and get a number slightly different from 7. Just keep rolling...) The result leads us to an important observation: The average value that we computed is simply the sum of each of the possible values of the random random variable (in this case the sum of the faces), multiplied by the probability of that value. In light of this, we make the following general definition:

Definition. Let X be a random variable with mass density $p(x)$. Recall that $p(x) = \mathbb{P}\{X = x\}$, the probability that the random variable X has the value x.

The **expectation** (or expected value) of X is denoted by $\mathbb{E}(X)$ and defined as

$$\mathbb{E}(X) = \sum_{\{x|p(x)>0\}} x \cdot p(x). \tag{1}$$

Thus, $\mathbb{E}(X)$ is the weighted average of the possible values of X (those are the x's in the formula) with the weights being the probabilities (those are the $p(x)$'s in the formula). This is exactly the computation that we did in the introductory example. The expectation of X is also called the **mean** or **average value** of X.

Example 1. Consider the experiment of flipping a coin three times . Let X be the random variable which gives the number of heads obtained in three flips. The possible values of X are $0, 1, 2$, and 3. In Example 1 of Probability #2, we determined that $p(0) = \frac{1}{8}$, $p(1) = \frac{3}{8}$, $p(2) = \frac{3}{8}$, and $p(3) = \frac{1}{8}$. Therefore,

$$\mathbb{E}(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 1.5$$

Notice that the expectation, $\mathbb{E}(X)$, need not be one of the values that X can have. Indeed, you couldn't flip a coin three times and get 1.5 heads! The fact that $\mathbb{E}(X) = 1.5$ says that if we do the experiment of flipping three coins many times, then we would expect about 1.5 heads per experiment. That is, on the average, about half the flips should be heads.

Example 2. Let Z be the payoff for playing the dice game in Example 2 of Probability #2. Then, Z has the possible values 2, 0, -1 and

$$\mathbb{E}(Z) = (-1) \cdot \frac{1}{2} + 0 \cdot \frac{1}{6} + 2 \cdot \frac{1}{3} = \frac{1}{6}.$$

Thus, on the average, you should expect to win $\frac{1}{6}$ of a dollar every time you play this game. So, if you play the game 60 times you could expect to be about \$10 dollars ahead.

Notice that all you need to compute $\mathbb{E}(X)$ is the probability mass density of X. If you have (or are given) the probability mass density you don't need to know the sample space.

Example 3. A certain slot machine that costs one dollar to play returns two dollars if you get three cherries and returns 15 dollars if you get three sevens. It gives you nothing for anything else. Suppose that the probability of getting three cherries is $\frac{1}{4}$ and the probability of getting three sevens is $\frac{1}{100}$. Since the sum of the probabilities is always one, the probability of "anything else" must be $\frac{74}{100}$. How much will you win or loss on the average if you play this game? Let Z be your profit. Z takes the value -1 with probability $\frac{74}{100}$, takes the value 1 with probability $\frac{1}{4}$ and takes the value 14 with probability $\frac{1}{100}$. Therefore,

$$\mathbb{E}(Z) \;=\; (-1)\cdot\frac{74}{100} \;+\; (1)\cdot\frac{25}{100} \;+\; (14)\cdot\frac{1}{100} \;=\; -\frac{35}{100}$$

Thus, you will lose on the average 35 cents per time that you play this game.

In the first three examples the random variable had only finitely many possible values. If X has infinitely many possible values, then formula (1) is an infinite series. In this case the mean, $\mathbb{E}(X)$, makes sense only if the infinite series converges. We will study infinite series in detail later.

Example 4. In Example 4 of Probability #2 we considered the experiment of flipping a coin until a head comes up. If we let X be the number of flips required, then X can take the values $1, 2, 3, 4, \ldots$, so X has infinitely many possible values. We showed there that

$$p(n) \;=\; \mathbb{P}(X = n) \;=\; \left(\frac{1}{2}\right)^n. \tag{2}$$

Thus, according to the definition (1),

$$\mathbb{E}(X) \;=\; 1\cdot\frac{1}{2} \;+\; 2\cdot\left(\frac{1}{2}\right)^2 \;+\; 3\cdot\left(\frac{1}{2}\right)^3 \;+\; 4\cdot\left(\frac{1}{2}\right)^4 \;+\; \cdots \tag{3}$$

$$\;=\; \frac{1}{2} \;+\; \frac{1}{2} \;+\; \frac{3}{8} \;+\; \frac{1}{4} \;+\; \cdots \tag{4}$$

Using techniques that we shall see later in the course one can show that this infinite series converges and that the sum is 2. Thus, on the average it should take two flips to get a head, which makes sense.

Example 5. An encyclopedia salesman visits three customers each day and with each he has a probability of $\frac{1}{4}$ of making a sale. For each sale he earns a commission of \$100 and if he makes three sales in day he earns a \$50 bonus from his company. What should he expect to earn each day? If the random variable X represents his daily earnings, then X takes the possible values \$0, \$100, \$200, and \$350 with probabilities $\mathbb{P}\{X = 0\} = \frac{27}{64}$, $\mathbb{P}\{X = 100\} = \frac{27}{64}$, $\mathbb{P}\{X = 200\} = \frac{9}{64}$, $\mathbb{P}\{X = 350\} = \frac{1}{64}$ (problem 5 of the last section). Therefore, his expected earning each day is:

$$\mathbb{E}(X) \;=\; 0\cdot\frac{27}{64} \;+\; 100\cdot\frac{27}{64} \;+\; 200\cdot\frac{9}{64} \;+\; 350\cdot\frac{1}{64} \;=\; 75.7812$$

Thus, he can expect to earn on the average \$76 each day.

Suppose, now, that the salesman is disgruntled and threatens to quit unless he can earn on the average \$90 per day. How should the company set his bonus (for selling 3 in one day) so that his expected earnings will be \$90? If the bonus is denoted by B, then his expected earnings will be:

$$\mathbb{E}(X) \;=\; 0\cdot\frac{27}{64} \;+\; 100\cdot\frac{27}{64} \;+\; 200\cdot\frac{9}{64} \;+\; (300+B)\cdot\frac{1}{64}$$

Setting his expected earnings equal to \$90 and solving for B yields B = \$960. Note that this is a huge bonus but the company is only going to be paying it $\frac{1}{64}$th of the time, so on the average the company will be paying him an extra \$14 per day approximately. That makes sense since \$90 - \$76 = \$14.

Example 6. Let us consider again the change of price of a single share of stock considered in Example 6 of Probability #2. We assumed that in each of two years the price would increase by one dollar, decrease by one dollar, or stay the same, each with probability $\frac{1}{3}$. We computed the probability mass density of X, the profit after two years, so it is easy to calculate the expected profit:

$$\mathbb{E}(X) \;=\; \frac{1}{9}(-2) \;+\; \frac{2}{9}(-1) \;+\; \frac{3}{9}(0) \;+\; \frac{2}{9}(1) \;+\; \frac{1}{9}(2) \;=\; 0.$$

This makes sense since the probability of going up by one unit each year is the same as the probability of going down by one unit.

Example 7. Here we reconsider Example 7 of Probability #2 in which the stock price rises by one dollar, decline by one dollar, or stays the same in each year, but the probabilities of these changes are not equal. In problem 7 of Probability #2, the student was asked to compute the probability mass density of the profit, X, after two years:

$$\mathbb{P}\{X=-2\}=\frac{1}{24} \quad \mathbb{P}\{X=-1\}=\frac{4}{24} \quad \mathbb{P}\{X=0\}=\frac{7}{24}$$
$$\mathbb{P}\{X=1\}=\frac{8}{24} \quad \mathbb{P}\{X=2\}=\frac{4}{24}$$

Therefore, the expected profit from holding this one share of this stock for the next two years is:

$$\mathbb{E}(X) \;=\; \frac{1}{24}(-2) \;+\; \frac{4}{24}(-1) \;+\; \frac{7}{24}(0) \;+\; \frac{8}{24}(1) \;+\; \frac{4}{24}(2) \;=\; \frac{10}{24}.$$

Example 8 (options). We saw in Example 6 that the expected profit after two years was 0. Suppose now that a brokerage firm offers me the right to buy one share next year at the current price. This is a simple example of an “option”.

What is this worth to me? Let's assume that my strategy is not to buy now. After one year, if the price has gone up then I will exercise my option and buy at the current price. If, after one year the price has remained the same or gone down, I do nothing. As before let X be the profit after two years. The market doesn't know about me and my choices, of course; so, as before, there are nine different possibles outcomes for price changes over the two years: $(+1,+1),(+1,0),\ldots,(-1,-1)$ each with probability $\frac{1}{9}$. For each outcome for which the change in the first year is -1 I do nothing, so my profit will be 0. For each outcome for which the change in the first year is 0 I do nothing, so again my profit will be 0. For each outcome for which the change in the first year is $+1$, I buy one share at the current price and thus make a profit of 0 (if the outcome is (+1, -1)), make a profit of 1 (if the outcome is (+1, 0)), or make a profit of 2 (if the outcome is (+1, +1)). Thus, my expected profit is:

$$\mathbb{E}(X) \;=\; \frac{7}{9}(0) \;+\; \frac{1}{9}(1) \;+\; \frac{1}{9}(2) \;=\; \frac{3}{9}.$$

Example 9 (options). Let us now do the same kind of calculation for Example 7, assuming again that a brokerage firm has offered me the right to buy next year at the current price. Let us assume that my strategy will be to do nothing now. After one year I will do nothing if the price has fallen, and I will purchase one share at the current price if the value of the share has either stayed the same or gone up during the first year. To compute the expectation of the profit we list all the outcomes, their probabilities, and the value of the profit X in each case.

outcome	probability	profit X
$(+1,+1)$	$\frac{4}{24}$	2
$(+1,0)$	$\frac{6}{24}$	1
$(+1,-1)$	$\frac{2}{24}$	0
$(0,+1)$	$\frac{2}{24}$	1
$(0,0)$	$\frac{3}{24}$	0
$(0,-1)$	$\frac{1}{24}$	-1
$(-1,+1)$	$\frac{2}{24}$	0
$(-1,0)$	$\frac{3}{24}$	0
$(-1,-1)$	$\frac{1}{24}$	0

Notice that the expected profit depends on the strategy I've adopted. Thus, the profit is 0 in the last three cases because my strategy is not to buy if the stock has gone down in the first year. From the table we can easily calculate the probability that X will take each of the values $-1,0,+1,+2$, and therefore we can calculate

$$\mathbb{E}(X) \;=\; \frac{1}{24}(-1) \;+\; \frac{11}{24}(0) \;+\; \frac{8}{24}(1) \;+\; \frac{4}{24}(2) \;=\; \frac{15}{24}.$$

Notice that this is a better expected return than $\frac{10}{24}$, which was the expected return if I buy one share now (calculated in Example 7). Thus, the option is worth $\frac{5}{24}$ to me.

Summary. To compute the expected value $\mathbb{E}(X)$ of a random variable X, one first computes the probability mass density of the random variable and then uses formula (1). Note that $\mathbb{E}(X)$ need not be a possible value of X. $\mathbb{E}(X)$ is also called the "mean" or "average value" of X.

Remark #1 (expected value is not guaranteed). It is important to keep in mind that $\mathbb{E}(X)$ is the average value one would expect after a large number of trials of the experiment (a large number of rolls in Example 2 or a large number of days selling encylclopedias in Example 5). In real life, the actual experiment may be performed only once. In Example 6 the stock price will change over the next two years only in one way. Thus, the meaning of X is that if, hypothetically, one was in this same situation many, many times (like the movie "Groundhog Day"), then we would expect an average return of $\mathbb{E}(X)$.

Remark #2 (how large is "large"). There is a deep mathematical question that we have swept under the rug in this discussion. We keep saying that after a large number of trials of an experiment, the average of values of X which we have obtained should be near $\mathbb{E}(X)$. How near? And, how many trials is a "large number"? These are deep questions that are considered in more advanced courses.

Remark #3 (guessing the future). Notice that our discussions of stock prices and options involved *assumptions* about the probabilities of different future events. Where do such assumptions (a less polite word would be "guesses") come from anyway? Well, sometimes they come from analyzing data from the past and sometimes they come from theories about the underlying mechanisms that produce the changes. Mathematicians have made important contributions to the development of such theories about the financial markets.

Problems.

(1) For each random variable in problems 1 – 4 of Probability #2, find $\mathbb{E}(X)$.

(2) Consider the following game. We choose a ball at random from an urn containing 7 red, 3 green, and 2 amber balls. We win \$2 if we choose a green ball, we lose \$1 if we choose a red ball, and we don't win or lose any money if we choose an amber ball. Let X represent our winnings. Find the probability mass density of X and the expected winnings, $\mathbb{E}(X)$, each time we play.

(3) A casino offers the following game. You pay \$8 to play. Then you roll a fair die twice and the casino pays you the sum of the faces. What is your expected payoff each time you play? Hint: The mass density for the sum of the faces was calculated in Example 3 of Probability #2.

(4) A casino offers the following game. They have the die from Example 2 of Probability #2 that has three red faces, two blue faces and one green face. You pay \$1 to play. Then you roll the die twice. If both rolls are the same, the casino pays you \$2. What is the casino's expected payoff everytime someone plays?

(5) Consider the following game played at a casino. A player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, for $i = 1, 2, 3$, then the player wins i dollars; on the other hand, if the number bet by the player does not appear on any of the dice, then the player loses 1 dollar. Is the game fair? If the game is not fair, who has the advantage, the player or the casino? Hint: Calculate $\mathbb{E}(X)$ where X equals the number of dollars the player wins or loses after each game.

(6) A casino offers the following game. They have the die from Example 2 of Probability #2 that has three red faces, two blue faces and one green face. You pay \$10 to play. Then you roll the die twice. If both rolls are red, the casino pays you \$10. If both rolls are blue, the casino pays you \$25 and if both rolls are green, the casino pays you \$100. What is the expected profit each time you play?

(7) Suppose that we have a biased coin which comes up tails $\frac{5}{8}$ of the time. Consider the experiment of flipping the coin until a head comes up and let X be the number of flips required. What values can X take? What is the probability mass density of X. Find an infinite series for the expected number of flips and use your hand calculator to estimate the sum of the series.

(8) You start a small insurance business on the side by providing car insurance for three of your neighbors. Each has a probability of $\frac{1}{10}$ of having an accident this year (assume no probability of each having more than one). For each accident you will have to pay out \$3000. How much should you charge for this coverage so that your expected profit for the year is \$1000?

(9) Find the expected profit for the purchase of one share of stock in the situation described in problem 8 of Probability #2.

(10) Find the expected profit for the purchase of one share of stock in the situation described in problem 9 of Probability #2.

(11) Find the expected profit for the purchase of one share of stock in the situation described in problem 10 of Probability #2.

(12) You are offered the option of buying next year at the current price one share of the stock described in problem 8 of Probability #2. Your strategy is to not purchase now but to purchase the share in one year if the price has not declined (in which case you do nothing). How much is this option worth to you?

(13) You are offered the option of buying next year at the current price one share of the stock described in problem 9 of Probability #2. Your strategy is to not purchase now but to purchase the share in one year if the price has not declined (in which case you do nothing). How much is this option worth to you?

(14) You are offered the option of buying next year at the current price one share of the stock described in problem 10 of Probability #2. Your strategy is to not purchase now but to purchase the share in one year if the price has not declined (in which case you do nothing). How much is this option worth to you?

Answers to Selected Problems.

1. 0, 1, 3, 0.
3. -1.
5. $-\frac{17}{216}$.
7. $\frac{3}{8}\{1 + 2(\frac{5}{8}) + 3(\frac{5}{8})^2 + 4(\frac{5}{8})^3 + \ldots\} \;=\; \frac{8}{3}$.
9. \$1.
11. $\$\frac{7}{3}$.
13. without option, $\mathbb{E}(X) = \$1$; with option, $\mathbb{E}(X) = \$\frac{18}{16}$.