Probability #2: Random Variables

In this lesson we introduce the concept of random variable, a central idea of probability theory. Consider an experiment which has a set, S, of possible outcomes. S is the sample space. A **random variable** is a function defined on S which takes values in the real numbers. That is, if X is the name of the random variable and s is a possible outcome of the experiment (that is, $s \epsilon S$), then $X(s)$ is a number.

Example 1. Consider the experiment of flipping a coin three times. The sample space of possible outcomes is

$$(H,H,H),\ (H,H,T),\ (H,T,H),\ (T,H,H)$$

$$(T,T,H),\ (T,H,T),\ (H,T,T),\ (T,T,T).$$

Let X be the random variable which assigns to each outcome the number of heads in the outcome. Then,

$$X((H,H,H)) = 3,\ \ X((H,H,T)) = 2,\ \ X((H,T,H)) = 2,\ X((T,H,H)) = 2$$

$$X((T,T,H)) = 1,\ \ X((T,H,T)) = 1,\ \ X((H,T,T)) = 1,\ \ X((T,T,T)) = 0.$$

Thus, X can take the values $0, 1, 2,$ and 3. How likely is it that X will have each of these values? Each outcome has probability $\frac{1}{8}$. There is only one outcome with 3 heads, so the probability that $X = 3$ is $\frac{1}{8}$. We write

$$\mathbb{P}(X = 3) \ = \ \frac{1}{8}.$$

Since there are three outcomes where $X = 2$, we have

$$\mathbb{P}(X = 2) \ = \ \frac{3}{8}.$$

Similarly, $\mathbb{P}(X = 1) \ = \ \frac{3}{8}$ and $\mathbb{P}(X = 0) \ = \ \frac{1}{8}$.

Here is another random variable on the same sample space. Let Y be the number of heads minus the number of tails. If we consult the list of possible outcomes above, we see that Y can take the values $3, 1, -1, \text{and} - 3$, and $\mathbb{P}(Y = 3) = \frac{1}{8}$, $\mathbb{P}(Y = 1) = \frac{3}{8}$, $\mathbb{P}(Y = -1) = \frac{3}{8}$, $\mathbb{P}(Y = -3) = \frac{1}{8}$.

Example 2. Suppose that we have a special die that is painted red on three sides, blue on two sides, and green on one side. We are going to play the following game. You roll the die and if red comes up, you pay me one dollar. If blue comes up you receive two dollars. If green comes up, no money changes hands. Let Z be the random variable which is the payoff to you after you roll. Then, Z takes on three values, $2, 0, \text{and} -1$ with the following probabilities:

$$\mathbb{P}(Z = 2) = \frac{1}{3}, \quad \mathbb{P}(Z = 0) = \frac{1}{6}, \quad \mathbb{P}(Z = -1) = \frac{1}{2}.$$

Since we will be very interested in the probabilities that a given random variable takes various values, it is convenient to define a function, $p(x)$, which has this information in it. Suppose that X is a random variable. If x is a possible value of X, we define

$$p(x) = \mathbb{P}(X = x).$$

If x is not a possible value of X, we define $p(x) = 0$. The function $p(x)$ is called the **mass density** of the random variable X.

Example 1 (again). If X is the number of heads in three flips, then, $p(0) = \frac{1}{8}$, $p(1) = \frac{3}{8}$, $p(2) = \frac{3}{8}$, $p(3) = \frac{1}{8}$. The value of the function p at any other x is 0.

Example 2 (again). Let $p(x)$ be the mass density for the payoff function Z in Example 2. Then, $p(-1) = \frac{1}{2}$, $p(0) = \frac{1}{6}$, $p(2) = \frac{1}{3}$. The value of the function p at any other x is 0.

It is sometimes helpful to graph the nonzero values of p with straight lines from the values down to the x-axis. Figures 1(a) and 1(b) graph the mass density functions for the random variables X of Example 1 and Z of Example 2. The graph shows visually which values have positive probability and the length of the line shows how much probability is associated with that value of x.

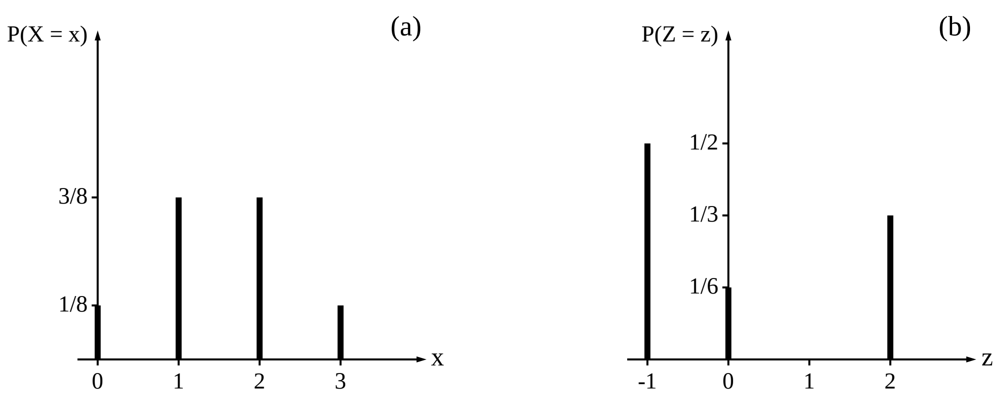

FIGURE 1

In Example 2 we see that the sum of the non-zero values of p is $p(-1) + p(0) + p(2) = 1$. In fact, we must always have the condition

$$(1) \qquad \sum_{\{x|p(x)>0\}} p(x) \;=\; 1$$

because the sum of the probabilities that X takes on all possible values is 1. The sum is over the set of numbers x such that $p(x) > 0$.

Example 3. Consider the experiment of rolling two dice. Let the random variable X be the sum of the numbers on the two faces showing. The possible outcomes of the experiment are shown in Table 1 of the last lesson. Each outcome has probability $\frac{1}{36}$. The random variable X takes the values $2, 3, 4, \ldots, 12$. There is only one outcome, $\{1, 1\}$, such that the sum of the faces is 2. Thus,

$$p(2) \;=\; \mathbb{P}(X = 2) \;=\; \frac{1}{36}.$$

There are two outcomes, $\{1, 2\}$ and $\{2, 1\}$, such that X has the value 3, so

$$p(3) \;=\; \mathbb{P}(X = 3) \;=\; \frac{2}{36}.$$

We can continue in this way and determine $p(k)$ for all the numbers $k = 2, 3, 4, \ldots 12$. The function p is largest when $k = 7$. There are six outcomes, $\{1, 6\}, \{2, 5\}, \{3, 4\}, \{4, 3\}, \{5, 2\}, \{6, 1\}$, for which $X = 7$, so

$$p(7) \;=\; \mathbb{P}(X = 7) \;=\; \frac{6}{36}.$$

The graph of the mass density function for X is shown in Figure 2. Check that the formula (1) is true by calculating that $p(2) + p(3) + \ldots + p(12) = 1$.

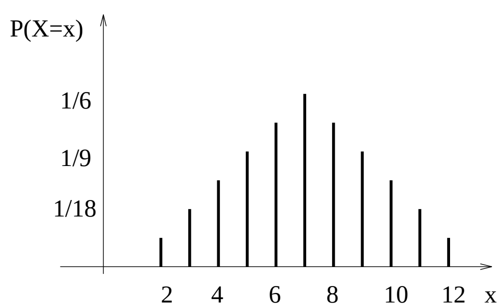

FIGURE 2

In all of our examples so far, the random variables could take only finitely many values. Here is an example of a random variable with infinitely many values.

Example 4. Consider the following experiment. We flip a coin until heads comes up. The sample space for this experiment is:

$$\{H, TH, TTH, TTTH, TTTTH, TTTTTH, \ldots\}$$

Notice that there are infinitely many different outcomes in the sample space. Let X be the number of flips required. The probability of getting a head on the first flip is $\frac{1}{2}$, so

$$p(1) \;=\; \mathbb{P}(X=1) \;=\; \frac{1}{2}.$$

If heads occurs for the first time on the second flip, that means that we got a tail on the first flip. The probability of flipping a coin twice and getting tails the first time and heads the second time is $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$. Thus,

$$p(2) \;=\; \mathbb{P}(X=2) \;=\; \frac{1}{4}.$$

If heads occurs for the first time on the third flip, that means that we got a tail on the first two flip. The probability of flipping a coin three times and getting tails the first two times and heads the third time time is $\left(\frac{1}{2}\right)^3$, so

$$p(3) \;=\; \mathbb{P}(X=3) \;=\; \frac{1}{8}.$$

Continuing in this way, we see that

$$p(n) \;=\; \mathbb{P}(X=n) \;=\; \left(\frac{1}{2}\right)^n. \tag{2}$$

Thus, X is a random variable whose value can be any positive integer and the probability that it takes the value n is $\left(\frac{1}{2}\right)^n$. In this case , condition (1) implies:

$$\tfrac{1}{2} \;+\; \left(\tfrac{1}{2}\right)^2 \;+\; \left(\tfrac{1}{2}\right)^3 \;+\; \left(\tfrac{1}{2}\right)^4 \;+\; \left(\tfrac{1}{2}\right)^5 \;+\; \ldots \;=\; 1.$$

The left-hand side is an infinite sum of positive numbers. Could this sum add up to something finite? We shall see later in the course that, indeed, infinite sums can sometimes make sense and add up to finite numbers. In particular, we shall see that the sum on the left, which is a special case of a geometric series, does add up to 1.

Example 5. Suppose that we consider the same experiment as in Example 4, except that this time we will assume that the coin is biased and comes up heads with probability q where $0<q<1$. Thus, the probability of getting tails is $1-q$. Again, let X be the number of flips required to get a head. Then,

$$p(1) \;=\; \mathbb{P}(X=1) \;=\; q,$$

If $X = 2$, then the first flip was a tail and the second flip a head, so

$$p(2) \;=\; \mathbb{P}(X = 2) \;=\; (1-q)q.$$

In general,

$$p(n) \;=\; \mathbb{P}(X = n) \;=\; (1-q)^{n-1}q,$$

Thus, to check (1) in this case we must verify

$$(3) \qquad q \;+\; (1-q)q \;+\; (1-q)^2 q \;+\; (1-q)^3 q \;+\; (1-q)^4 q \;+\; \ldots \;=\; 1.$$

If we factor q out, the left-hand side of (3) is:

$$q\{1 \;+\; (1-q) \;+\; (1-q)^2 \;+\; (1-q)^3 \;+\; (1-q)^4 + \ldots\}$$

If we set $\alpha = (1-q)$, the expression in brackets can be written

$$1 \;+\; \alpha \;+\; \alpha^2 \;+\; \alpha^3 \;+\; \alpha^4 + \ldots$$

This infinite sum is called the geometric series and we shall see later that if $|\alpha| < 1$, then the sum makes sense and equals $\frac{1}{1-\alpha}$. Since $\alpha = (1-q)$, the left-hand side of (3) is

$$q \cdot \frac{1}{1-\alpha} \;=\; q \cdot \frac{1}{1-(1-q)} \;=\; q \cdot \frac{1}{q} \;=\; 1$$

Thus, we have checked that (1) holds.

Example 6. Let us suppose that you own one share of stock for two years and that each year the value of the stock either rises by one dollar (probability $= \frac{1}{3}$), stays the same (probability $= \frac{1}{3}$), or declines by one dollar (probability $= \frac{1}{3}$). Suppose that the change in value in the first year is independent of the change in value in year the second year. Let X be the profit after two years. What is the probability density of X?

The sample space is the set of pairs (c_1, c_2) where c_1 is the change in the first year and c_2 is the change in the second year. That is, c_1 is either $+1$ or 0 or -1 and similarly for c_2. Thus, there are nine possible outcomes, $(-1,-1), (-1,0), \ldots$. The probability of a decline of one dollar in the first year is $\frac{1}{3}$ and the probability of a decline of one dollar in the second year is $\frac{1}{3}$. Since we are assuming that the changes in the two years are independent, we calculate that the probability of the outcome $(-1,-1)$ (i.e. two declines) is $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$. Similarly, we see that the probability of each of the nine outcomes (c_1, c_2) is $\frac{1}{9}$.

The profit X is $c_1 + c_2$, that is, the total change in value after two years. Since c_1 and c_2 are each $+1$ or 0 or -1, the possible values of X are $-2, -1, 0, +1, +2$. There is only one outcome, namely $(-1,-1)$, such that $X = -2$ so,

$$p(-2) \;=\; \mathbb{P}\{X = -2\} \;=\; \frac{1}{9}.$$

However, notice that there are two outcomes in the sample space, $(-1, 0)$ and $(0, -1)$ such that $X = -1$. Thus,

$$p(-1) \;=\; \mathbb{P}\{X = -1\} \;=\; \frac{2}{9}.$$

And, there are three outcomes in the sample space, $(-1, +1), (0, 0), \text{and}(+1, -1)$ such that $X = 0$. Thus,

$$p(0) \;=\; \mathbb{P}\{X = 0\} \;=\; \frac{3}{9}.$$

Similarly, we see that

$$p(1) \;=\; \mathbb{P}\{X = 1\} \;=\; \frac{2}{9}.$$

and

$$p(2) \;=\; \mathbb{P}\{X = 2\} \;=\; \frac{1}{9}.$$

Example 7. As in Example 6, we suppose that you own one share of stock for two years and that each year the value of the stock either rises by one dollar, stays the same, or declines by one dollar and the changes in the two years are independent. However, we assume that in the first year, the change $+1$ has probability $\frac{1}{2}$, the change 0 has probability $\frac{1}{4}$ and the change -1 has probability $\frac{1}{4}$. In the second year, we assume that the change $+1$ has probability $\frac{1}{3}$, the change 0 has probability $\frac{1}{2}$ and the change -1 has probability $\frac{1}{6}$. Find the probability density of the profit, X, after two years. The sample space is the same as in Example 6 and, as before, X has the possible values $-2, -1, 0, +1, +2$. However, now the different outcomes in the sample space have different probabilities. For example, the probability of the outcome $(-1, -1)$ is $\frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}$. Once one knows the probabilities of all the outcomes, one can find the probability that $X = n$ by adding the probabilities of all outcomes (c_1, c_2) such that $c_1 + c_2 = n$. The student is asked to do this in problem 7.

Problems.

(1) Consider the experiment of flipping a fair coin twice. Let X be the number of heads minus the number of tails. What are the possible values of X? Find and graph the probability mass density of X.

(2) Consider the experiment of flipping a biased coin (which comes up heads with probability $= \frac{3}{4}$) twice. Let X be the number of heads minus the number of tails. What are the possible values of X? Find and graph the probability mass density of X.

(3) Consider the experiment of flipping a fair coin three times. Let X be the square of the number of heads. What are the possible values of X? Find and graph the probability mass density of X.

(4) Consider the experiment of flipping a fair coin three times. Let X be the square of the number of heads minus two times the number of tails. What are the possible values of X? Find and graph the probability mass density of X.

(5) An encyclopedia salesman visits three customers each day and with each he has a probability of $\frac{1}{4}$ of making a sale. For each sale he earns a commission of \$100 and if he makes three sales in day he earns a \$50 bonus from his company. Let X be his daily earnings. What are the possible values of X? What is the mass density of X?

(6) Consider the following game. We choose a ball at random from an urn containing 7 red, 3 green, and 2 amber balls. We win \$2 if we choose a green ball, we lose \$1 if we choose a red ball, and we don't win or lose any money if we choose an amber ball. Let X represent our winnings. Find the probability the mass density of X.

(7) Complete the calculations in Example 7 by finding the probability of each of the nine outcomes. Then find the probability mass density of X, the profit.

(8) You own one share of stock for three years and each year the value of the stock either rises by one dollar (probability $= \frac{2}{3}$) or declines by one dollar (probability $= \frac{1}{3}$); it never stays the same. Suppose that the yearly changes are independent of each other. What is the sample space? What is the probability of each outcome in the sample space? Let X be the profit after three years. What values can X have? What is the probability mass density of X?

(9) You own one share of stock for two years and each year the value of the stock changes by $+2, +1, 0, -1$ each with probability $\frac{1}{4}$. Suppose that the changes in the two years are independent. What is the sample space? What is the probability of each outcome in the sample space? Let X be the profit after two years. What values can X have? What is the probability mass density of X?

(10) You own one share of stock for two years. The first year the value of the stock changes by $+1, 0, -1$, with probabilities $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, respectively. The second year the value of the stock changes by $+1, +2, +3$, with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, respectively. Suppose that the changes in the two years are independent. What is the sample space? What is the probability of each outcome in the sample space? Let X be the profit after two years. What values can X have? What is the probability mass density of X?

Answers to Selected Problems.

1. $\mathbb{P}\{X=2\}=\frac{1}{4}$, $\mathbb{P}\{X=0\}=\frac{1}{2}$, $\mathbb{P}\{X=-2\}=\frac{1}{4}$.
3. $\mathbb{P}\{X=0\}=\frac{1}{8}$, $\mathbb{P}\{X=1\}=\frac{3}{8}$, $\mathbb{P}\{X=4\}=\frac{3}{8}$ $\mathbb{P}\{X=9\}=\frac{1}{8}$.
5. $\mathbb{P}\{X=0\}=\frac{27}{64}$, $\mathbb{P}\{X=100\}=\frac{27}{64}$, $\mathbb{P}\{X=200\}=\frac{9}{64}$, $\mathbb{P}\{X=350\}=\frac{1}{64}$.
7. $\mathbb{P}\{X=-2\}=\frac{1}{24}$, $\mathbb{P}\{X=-1\}=\frac{4}{24}$, $\mathbb{P}\{X=0\}=\frac{7}{24}$, $\mathbb{P}\{X=+1\}=\frac{8}{24}$, $\mathbb{P}\{X=+2\}=\frac{4}{24}$.
9. $\mathbb{P}\{X=4\}=\frac{1}{16}$, $\mathbb{P}\{X=3\}=\frac{2}{16}$, $\mathbb{P}\{X=2\}=\frac{3}{16}$, $\mathbb{P}\{X=1\}=\frac{4}{16}$, $\mathbb{P}\{X=0\}=\frac{3}{16}$, $\mathbb{P}\{X=-1\}=\frac{2}{16}$, $\mathbb{P}\{X=-2\}=\frac{1}{16}$.