Probability #1: Outcomes and Events

Games of chance, such as those involving dice, have been played for over 5,000 years. Ever since 1550 A.D., mathematicians have been involved with calculating the probabilities or chances of winning at gambling. For instance, in approximately 1620, Galileo wrote a paper on dice probabilities. The year 1654 is often thought of as the beginning of probability theory since it was at that time that Blaise Pascal and Pierre de Fermat began correspondence on the subject. Pierre Simon, the Marquis de Laplace, was an important contributer to probability theory; in 1812 he proved the central limit theorem which provides an explanation as to why so many data sets follow a distribution that is bell-shaped, i.e., normally distributed. We shall explore this idea later in the course during the normal distribution lab. In Laplace's book *Analytical Theory of Probability* he says:

"*We see that the theory of probability is at bottom only common sense reduced to calculation; it makes us appreciate with exactitude what reasonable minds feel by a sort of instinct, often without being able to account for it ... It is remarkable that this science, which originated in the consideration of games of chance, should become the most important object of human knowledge. ... The most important questions in life are, for the most part, really only problems of probability.*"

While Laplace may have overstated the case, probability theory is certainly a very important modeling tool in quantum mechanics, in the financial markets, and in biology.

Probability is an extension of the idea of a proportion, or the ratio of a part to the whole. Suppose that there are 6 green balls and 4 red balls in a urn. You mix them well and then reach in without looking and pull one out. Since 60% of the balls are green, we expect "on the average" that about 60% of the time we will get a green ball. Intuitively, here is what the phrase "on the average" means. After noting the color of the ball, we put it back, mix well and select again. Let N_g denote the number of green balls that we have obtained after performing this experiment N times. Then, we expect the ratio N_g/N to get close to .6 as N increases and similarly, the probability of getting a red ball is .4. Selecting a ball is called a "random event" because the outcome is uncertain.

An **experiment** is a well-defined procedure. The set of possible outcomes of the experiment is called the **sample space**.

So, in our example above the experiment is putting 6 green balls and 4 red balls in an urn, mixing well, and selecting one ball at random (without looking). The sample space has two elements in it, G (for green) and R (for red).

Example 1. Consider the experiment of rolling a single die. There are six possible outcomes: one of the numbers 1, 2, 3, 4, 5, or 6. In this case, the sample space has six outcomes. Assuming that we have a fair die (that is, each side is equally likely to turn up), we say that the probability of each of these outcomes is 1/6.

An **event** is a subset of the sample space of possible outcomes. If A is an event, we denote by $\mathbb{P}(A)$ the probability that the event will happen.

In the case of rolling a die, an event is a subset of the set of six integers $\{1,2,3,4,5,6\}$. Suppose that A is the event that the die shows an even number. That is, $A = \{2,4,6\}$. Then

$$\mathbb{P}(A) \;=\; \mathbb{P}(\{2,4,6\}) \;=\;=\; \frac{3}{6},$$

since the event $\{2,4,6\}$ contains 3 out of the 6 equally likely outcomes. Similarly, suppose that B is the event that the outcome shows a number less than 3. Then, $B = \{1,2\}$ and $\mathbb{P}(B) = \frac{1}{3}$.

There are two fundamental principles to remember. **The probability of an outcome or an event is always a number between zero and one** $(0 \leq p \leq 1)$, because it is the proportion of repeated experiments (on the average) in which the outcome or event occurs. Second, **the sum of the probabilities of the possible outcomes of an experiment must add up to one**; that is, one of the outcomes must occur.

Example 2. Consider the experiment of flipping a coin three times. If we denote a head by H and a tail by T, we can list the eight possible outcomes:

$$(H,H,H),\ (H,H,T),\ (H,T,H),\ (T,H,H)$$

$$(T,T,H),\ (T,H,T),\ (H,T,T),\ (T,T,T)$$

each of which occurs with probability $\frac{1}{8}$. To calculate the probability of any particular event (subset of the sample space of outcomes), we add the probabilities of the outcomes in the event set. Thus,

(a) The probability that all three flips are heads $= \frac{1}{8}$.
(b) The probability that exactly two flips are heads $= \frac{3}{8}$.
(c) The probability that the first flip is tails $= \frac{1}{2}$.
(d) The probability that at least one flip is heads $= \frac{7}{8}$.
(e) The probability that heads and tails alternate $= \frac{1}{4}$.

Adding probabilities: Suppose that we are considering two events A and B and we would like to know the probability of either A or B happening? Is it $\mathbb{P}(A) + \mathbb{P}(B)$? Let's look at another example.

Example 3. Suppose that we roll a die twice. We denote the outcome of the first roll by X and the outcome of the second roll by Y. Suppose we want to find

$$\mathbb{P}(X \geq 3 \textbf{ or } Y \geq 2).$$

This is the probability of either getting a 3, 4, 5, or 6 on the first roll, or getting a 2, 3, 4, 5, or 6 on the second roll. When we say (A **or** B) we mean either A or B or both. The probability that X is greater than or equal to three is 4/6 and the probability that Y is greater than or equal to two is 5/6. If we add these probabilities we get 9/6, which should set off alarm bells since we cannot have a probability greater than one. Let's figure out what is going on here.

	1	2	3	4	5	6	Y
1	1,1	1,2	1,3	1,4	1,5	1,6	
2	2,1	2,2	2,3	2,4	2,5	2,6	
3	3,1	3,2	3,3	3,4	3,5	3,6	
4	4,1	4,2	4,3	4,4	4,5	4,6	
5	5,1	5,2	5,3	5,4	5,5	5,6	
6	6,1	6,2	6,3	6,4	6,5	6,6	
X							

TABLE 1. Outcomes of rolling a die twice

First, the only time when ($X \geq 3$ or $Y \geq 2$) is not true is when $X = 1$ and $Y = 1$ or when $X = 2$ and $Y = 1$. Thus there are only two events out of 36 possible events where our condition is not satisfied; therefore, the correct probability is $(36-2)/36 = 34/36$. The condition $X \geq 3$ (which has probability 4/6) corresponds to all the events in the bottom 4 rows of Table **??**. The condition $Y \geq 2$ (probability 5/6) corresponds to all events in the 5 right - hand columns of Table **??**. When we add the probability of $X \geq 3$ to the probability of $Y \geq 2$ we are double counting all of the events which are both in the bottom 4 rows and in the 5 right-hand columns! Notice that there are $5 \cdot 4 = 20$ different outcomes in the intersection of the bottom 4 rows and the 5 right-hand columns. Thus, we need to subtract the probability of the outcomes in the intersection, $\frac{20}{36}$. Notice that the intersection of the bottom 4 rows and the 5 right-hand rows corresponds to the event ($X \geq 3$ and

$Y \geq 2)$. Thus, we have

$$\begin{aligned}
\mathbb{P}(X \geq 3 \text{ or } Y \geq 2) &= \mathbb{P}(X \geq 3) + \mathbb{P}(Y \geq 2) - \mathbb{P}(X \geq 3 \text{ and } Y \geq 2) \\
&= \frac{4}{6} + \frac{5}{6} - \frac{20}{36} \\
&= \frac{34}{36}.
\end{aligned}$$

In general, we have the following rule:

Addition Rule: $\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \text{ and } B)$.

Thus, the probability of the event $(A \text{ or } B)$ is equal to the probability of A plus the probability of B minus the probability of $(A \text{ and } B)$, see Figure **??**.

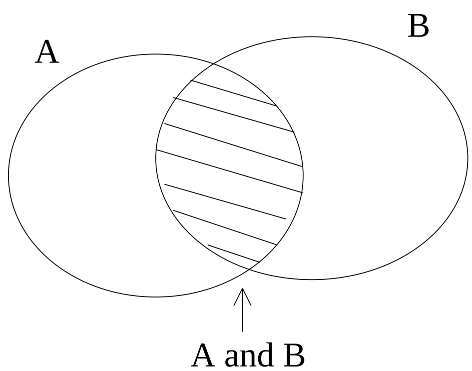

FIGURE 1. Addition Rule

Note that in terms of sets the event $(A \text{ or } B)$ corresponds to the set $A \cup B$ and the event $(\ A \text{ and } B)$ corresponds to the set $A \cap B$. Thus, we can rewrite the above rule as

Addition Rule: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Example 4: A card is selected at random from a deck of 52 cards. We wish to find the probability that the card selected will be a Queen or a diamond. Letting Q represent the event that a Queen is selected and D represent the event that a diamond is selected, and using the addition rule, we have

$$\begin{aligned}\mathbb{P}(Q \text{ or } D) &= \mathbb{P}(Q) + \mathbb{P}(D) - \mathbb{P}(Q \text{ and } D) \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \\ &= \frac{4}{13}.\end{aligned}$$

We can confirm this computation by noting that a deck of cards has 13 diamonds and 3 queens which are not diamonds; thus $\frac{16}{52} = \frac{4}{13}$ of the cards we are after are either queens or diamonds.

Independence: Suppose that we are considering the outcomes of a particular experiment. If A and B are events (subsets of the sample space of outcomes), we say that A and B are **independent** if

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A)\mathbb{P}(B). \tag{1}$$

As we shall see in the following example, A and B are independent if knowing that A is true (that is, that the outcome is in A) does not affect the probability that B is true and vice-versa. This is why we use the word "independent."

Example 5. Consider the experiment in Example 2 where we flipped a coin 3 times. Let A be the event that a head comes up on the first flip. There are 4 outcomes in the sample space for which this is true – each with probability $\frac{1}{8}$. Therefore, $\mathbb{P}(A) = \frac{1}{2}$. Let B be the event that a head comes up on the third flip. Again, we calculate easily that $\mathbb{P}(B) = \frac{1}{2}$. What is the probability of (A and B)? This is the subset $A \cap B$ of outcomes such that there is a head on the first flip *and* a head on the third flip. There are two such outcomes, (H, T, H) and (H, H, H), each with probability $\frac{1}{8}$. Thus, $\mathbb{P}(A \text{ and } B) = \frac{1}{4}$. Therefore,

$$\mathbb{P}(A)\mathbb{P}(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = \mathbb{P}(A \text{ and } B).$$

So, by definition, the events A and B are independent. This makes sense because the outcome of the first flip should not affect the probability of the outcome of the third flip. To see an example of events that are not independent, let A again denote the event that the first flip is a head. Let C denote the event that at least two of the three flips are heads. Looking at the list of outcomes, we see that in four of them there are two or more heads. Thus, $\mathbb{P}(C) = \frac{1}{2}$. The event ($A$ and C) is the set of outcomes in which the first flip is a head *and* at least two out of the three are heads. Consulting the list of possible outcomes we see that there are three outcomes for which this is true. Thus $\mathbb{P}(A \text{ and } C) = \frac{3}{8}$. Since

$$\mathbb{P}(A)\mathbb{P}(C) \;=\; \frac{1}{2}\cdot\frac{1}{2} \;\neq\; \frac{3}{8} \;=\; \mathbb{P}(A \text{ and } C),$$

we see that A and C are not independent. This makes sense because getting a head on the first flip surely affects the probability that we will get at least two heads in the three flips.

Independence can sometimes be used to make calculations easier.

Example 6. Suppose that we roll a die three times. Each time we roll there are 6 possible results for that roll, so the total number of outcomes is 6^3. Therefore, the probability of any particular outcome, say $(5,1,4)$, is $\frac{1}{6^3}$. Let A be the event that we get a 4 on the second roll and not a 4 on both the first and third rolls. Let B_1 be the event that we do not get a 4 on the first roll; then $\mathbb{P}(B_1) = \frac{5}{6}$. Let B_2 be the event that we do get a 4 on the second roll; then $\mathbb{P}(B_2) = \frac{1}{6}$. Let B_3 be the event that we do not get a 4 on the third roll. Then $\mathbb{P}(B_3) = \frac{5}{6}$. Now, $A = (B_1 \text{ and } B_2 \text{ and } B_3)$. Therefore, if B_1, B_2, and B_3 are all independent (as we believe they are), then the probability of A should just be the product of the probabilities of B_1, B_2, and B_3. That is,

$$\mathbb{P}(A) \;=\; \mathbb{P}(B_1)\mathbb{P}(B_2)\mathbb{P}(B_3) \;=\; \frac{5}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}.$$

Now, let's use these ideas to calculate the probability that if we roll a die three times we will get exactly one 4. This event could happen in three ways. We could get a 4 the first roll and then two non-4s; call this the event C_1. We could get a non-4, a 4, and then a non-4; this is just the event A above. Finally, we could get two non-4s, and then a 4; call this the event C_3. Each of these three events has probability

$$\left(\frac{1}{6}\right)\cdot\left(\frac{5}{6}\right)^2$$

so,

$$\mathbb{P}(C_1) \;+\; \mathbb{P}(A) \;+\; \mathbb{P}(C_3) = \;3\left(\frac{1}{6}\right)\cdot\left(\frac{5}{6}\right)^2$$

We can add the probabilities here because if there is exactly one 4 it could be on the first roll or the second but not both; that is, $\mathbb{P}(C_1 \cap A) = 0$. Similarly, it could be on the second or the third roll but not both; that is, $\mathbb{P}(A \cap C_3) = 0$.

Note to Students. You may be wondering why Math 32L starts with three lessons on probability theory. There are several reasons. First, these lessons develop intuition which we shall use in the ninth week of this course when we study applications of calculus to probability theory. Second, certain infinite sums arise naturally in probability theory. The need to compute infinite sums motivates our next topic, which is the study of infinite series. Finally, probability is a beautiful and important subject (in both pure and applied mathematics), and you should know something about it!

Problems.

(1) Suppose that a bag contains 7 black balls, 6 yellow balls, 4 green balls, and 3 red balls. You shake the bag well, and remove one ball without looking into the bag.
   (a) What is the probability that the ball you remove is red? black? yellow? green? white?
   (b) What is the probability that the ball you pick is either black or green?
   (c) What is the probability that you have picked a ball whose color is not red?

(2) A die is painted so that three sides are red, two sides are blue and one side is green. Thus, rolling the die has three possible outcomes R, B, and G.
   (a) What is the probability the die will come up blue?
   (b) What is the probability the die will not come up red?
   (c) What is the probability that the face showing is either red or blue?

(3) Suppose that we flip a fair coin four times.
   (a) How many outcomes are in the sample space?
   (b) What is the probability that all four flips are tails?
   (c) What is the probability that there will be one or fewer heads?
   (d) What is the probability that there will be equal numbers of heads and tails?

(4) The painted die from problem 2 is rolled twice. Denote the nine possible outcomes by RR, RB, etc..
   (a) Find the probability of each element of the sample space.
   (b) What is the probability that at least one roll will be red?
   (c) What is the probability that neither roll is blue?
   (d) What is the probability that the two rolls will have different colors?

(5) If we roll a fair die, what is the probability that after six rolls we:
   (a) do not get a 6?
   (b) get a 6 on the first roll, but not after?
   (c) get exactly one 6?

(6) The painted die from problem 2 is rolled twice. Use the addition rule to find the following probabilities.
   (a) The probability that either both rolls are red or both rolls are blue.

(b) The probability that either both rolls are red or exactly one roll is blue.
(c) The probability that either at least one roll is red or exactly one one roll is blue.
(d) The probability that either at least one roll is red or at least one roll is blue.

(7) Suppose that a fair coin is flipped three times.
(a) Let A be the event that the first flip is heads. Let B be the event that the second and third flips are the same. Find $\mathbb{P}(A \text{ and } B)$, and prove that A and B are independent.
(b) Let A be the event that the first flip is heads. Let B be the event that at least two flips are heads. Find $\mathbb{P}(A \text{ and } B)$ and prove that A and B are not independent. Why does this make sense?

(8) Suppose that a fair die is rolled twice.
(a) Let A be the event that the first roll is ≥ 2. Let B be the event that the second roll is ≤ 4. Find $\mathbb{P}(A \text{ and } B)$ and prove that A and B are independent.
(b) Let A be the event that the first roll is ≥ 2. Let B be the event that the sum of the rolls is ≤ 4. Find $\mathbb{P}(A \text{ and } B)$, and prove that A and B are not independent. Why does this make sense?

Selected Answers.

1. (a) 3/20, 7/20, 6/20, 4/20, 0; (b) 11/20; (c) 17/20.
3. (a) 16; (b) $\left(\frac{1}{2}\right)^4$; (c) $\frac{5}{16}$; (d) $\frac{3}{8}$.
5. (a) $\left(\frac{5}{6}\right)^6$; (b) $\left(\frac{5}{6}\right)^5 \frac{1}{6}$; (c) $6\left(\frac{5}{6}\right)^5 \frac{1}{6}$.
7. (a) $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(A \text{ and } B) = \frac{1}{4}$.
   (b) $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(A \text{ and } B) = \frac{3}{8}$.