Integrating to Infinity

Part I: The basic ideas

1. Compute by hand each of the following integrals and use your calculator to approximate the answer to six decimal places:

$$\int_1^2 e^{-t}dt = 0.2325441579 \qquad \int_1^{10} e^{-t}dt = 0.3678340412$$

$$\int_1^{100} e^{-t}dt = 0.3678794412 \qquad \int_1^{1000} e^{-t}dt = 0.3678794412$$

Considering both the values above and the fact that $\int_1^{N} e^{-t}dt = -e^{N} + e^{-1}$ we deduce that $\int_1^{N} e^{-t}dt$ approaches e^{-1} as $N \to \infty$; i.e., the value of $\int_1^{\infty} e^{-t}dt$ ought to be $e^{-1} = 0.36787944124\ldots$.

2. Compute by hand each of the following integrals and use your calculator to approximate the answer to three decimal places.

$$\int_0^2 \frac{2}{1+x^2}\,dx = 2\,arctan(2) - 2\,arctan(0) = 2\,arctan(2) \approx 2.214297$$

$$\int_0^b \frac{2}{1+x^2}\,dx = 2\,arctan(b) - 2\,arctan(0) = 2\,arctan(b)$$

$$\lim_{b\to\infty} \int_0^b \frac{2}{1+x^2}\,dx = \lim_{b\to\infty}[2\,arctan(b) - 2\,arctan(0)]$$

$$= \lim_{b\to\infty} 2\,arctan(b) = 2 \cdot \frac{\pi}{2} = \pi$$

We define $\int_0^{\infty} \frac{2}{1+x^2}\,dx$ to be the limit computed in the last step. If we measure the area under the curve $\frac{2}{1+x^2}$ from 0 farther and farther toward the right, the measure of the area clearly approaches π.

Part II: Some computations

1. Consider $\int_1^\infty \frac{1}{t}\, dt$.

$$\int_1^\infty \frac{1}{t}\, dt = \lim_{b\to\infty} \int_1^b \frac{1}{t}\, dt = \lim_{b\to\infty} [ln(b) - ln(1)] = \lim_{b\to\infty} ln(b) = \infty\ .$$ The integral diverges.

2. Does the integral $\int_1^\infty \frac{1}{t^2}\, dt$ converge?

$$\int_1^\infty \frac{1}{t^2}\, dt = \lim_{b\to\infty} \int_1^b \frac{1}{t^2} dt = \lim_{b\to\infty} -t^{-1}\Big]_1^b = \lim_{b\to\infty} [-b^{-1} - (-1^{-1})] = 1.$$ The integral converges.

3. For which values of the constant p does the integral $\int_1^\infty \frac{1}{t^p}\, dt$ converge?

Case $p \neq 1$: $\int_1^\infty \frac{1}{t^p}\, dt = \lim_{b\to\infty} \int_1^b t^{-p}\, dt = \lim_{b\to\infty} \frac{1}{1-p}\, t^{1-p}\Big]_1^b = \lim_{b\to\infty} \left[\frac{1}{1-p} b^{1-p} - \frac{1}{1-p}(1)\right]$

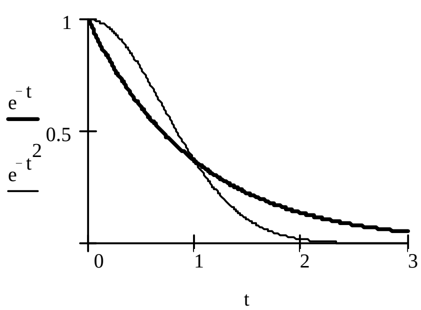
If $p > 1$, then b^{1-p} approaches 0, and the integral converges to $\frac{1}{p-1}$.

If $p < 1$, then b^{1-p} increases without bound and the integral diverges.

Case $p = 1$: The integral diverges as shown in step 1 above.

Part III: Extending these ideas

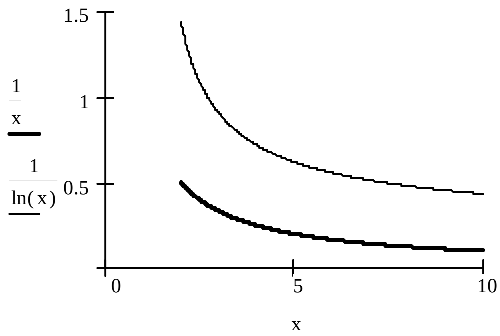
1. Graph $y = e^{-t}$ and $y = e^{-t^2}$ on the same set of axes for $t \geq 1$. Use an argument with areas to determine whether $\int_1^\infty e^{-t^2} dt$ converges.



Because the curve e^{-t} lies above the curve e^{-t^2} for $t \geq 1$, the area under e^{-t} contains the area under e^{-t^2} for $t \geq 1$. Because the area under e^{-t} is finite, it follows that the area under e^{-t^2} must also be finite.

We cannot compute $\int_1^\infty e^{-t^2} dt$ using the Fundamental Theorem of Calculus, because e^{-t^2} has no elementary antiderivative.

2. Graph $y = \frac{1}{ln\,x}$ and $y = \frac{1}{x}$ on the same set of axes using $[2,\infty]$ as the common domain. Use that graph to decide whether $\int_2^\infty \frac{1}{ln\,x}\,dx$ converges.



The curve $\frac{1}{ln\,x}$ lies above the curve $\frac{1}{x}$ for $x \geq 2$; thus, the area under $\frac{1}{ln\,x}$ is larger than the area under $\frac{1}{x}$. We've shown that the area under $\frac{1}{x}$ is not bounded; thus, the area under $\frac{1}{ln\,x}$ is also unbounded. We deduce that the integral $\int_2^\infty \frac{1}{ln\,x}$ diverges.

3. Assume that $0 \leq f(t) \leq g(t)$ for $t \geq a$. Exactly as we did in steps 1 and 2 above, we deduce the following two statements:

If $\int_a^\infty g(\mathrm{t})\,dt$ converges, then $\int_a^\infty f(t)\,dt$ must also converge.

If $\int_a^\infty f(t)\,dt$ diverges, then $\int_a^\infty g(\mathrm{t})\,dt$ must also diverge.

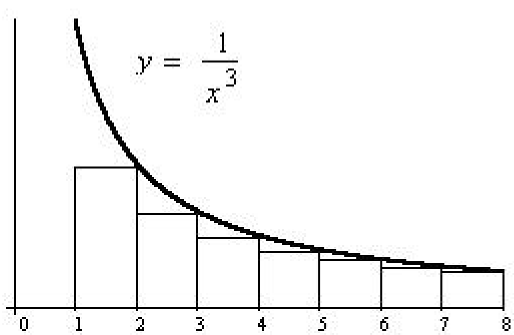
Note: Be sure students understand the following:

If $\int_a^\infty g(\mathrm{t})\,dt$ diverges, then we can deduce no additional information.

If $\int_a^\infty f(t)\,dt$ converges, then we can deduce no additional information.

Part IV: A connection to series

Draw a graph of $\frac{1}{x^3}$ from 1 to ∞. On the same graph, draw a picture that represents $\sum_{k=2}^{\infty} \frac{1}{k^3}$ as an area. What can you conclude about the convergence of $\sum_{k=2}^{\infty} \frac{1}{k^3}$?



The area of the first rectangle is the first term of the series. The area of the second rectangle is the second term of the series, and so on. The area under the curve contains all of the rectangles; thus, the sum of the areas of the rectangles must be finite because $\int_1^\infty \frac{1}{x^3}\,dx$ is finite. In other words the series must converge.

Part V: Conclusion

In the preview we raised a question about the area under the function $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$ and for some positive constant λ. The area under the graph of this function over an interval [a,b] is the fraction of certain electrical components which we expect to fail between time a and time b.

1. Compute the total area under f for $t \geq 0$. Why does the answer you get make sense in terms of probability and the context of the problem?

$$\int_0^\infty \lambda e^{-\lambda t}\,dt = \lim_{b\to\infty} \int_0^b \lambda e^{-\lambda t}\,dt = \lim_{b\to\infty} -e^{-\lambda t}\Big]_0^b = \lim_{b\to\infty} \left(-e^{-\lambda b} + e^0\right) = 1\ .$$

The probability that all components will eventually fail is 1, just as it should be.

2-3. Let T be a fixed value of t. $[T > 0]$ Compute each of the following integrals.

The fraction of components which last at least until time T is the area under the curve to the right of T:

$$\int_T^\infty \lambda e^{-\lambda t}\,dt = \lim_{b\to\infty} \int_T^b \lambda e^{-\lambda t}\,dt = \lim_{b\to\infty} -e^{-\lambda t}\Big]_T^b = \lim_{b\to\infty} \left(-e^{-\lambda b} + e^{-\lambda T}\right) = e^{-\lambda T}$$

The fraction of components which fail before time T is the area under the curve from time 0 to time T:

$$\int_0^T \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big]_0^T = -e^{-\lambda T} + e^0 = -e^{-\lambda T} + 1$$

Be sure students check that the sum of the two probabilities above is equal to 1, as it should be.