Integrating to Infinity

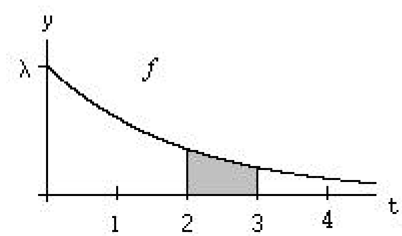
Purpose:

We will introduce the idea of a definite integral with no upper bound on the interval of integration. We will see that such integrals arise from natural questions and that they are also useful in examining the convergence or divergence of some types of infinite series.

Preview

There is a probability model that describes the longevity of electronic devices, such as light bulbs or computer disk drives. Statisticians use a function of the form $f(t) = \lambda e^{-\lambda t}$, where $t \geq 0$ and λ is a positive constant which depends upon the particular electrical device. This function f has the property[1] that the area under the graph of f over an interval [a,b] is the fraction of these electrical components which will fail between time a and time b.

Thus, if t is measured in years, $\int_2^3 \lambda e^{-\lambda t}\, dt$ would tell us what fraction of these components will "die" in the third year of use.



A simple question leads to a not-so-simple mathematical issue: what fraction of these components will last at least 10 years? Could that be the area under f from 10 to infinity, whatever that means? Or could we simply say the fraction of the components which last at least 10 years is

$$1 - \text{(the fraction which fail within 10 years)} \ ?$$

Indeed, to be consistent with our previous work in probability, this last idea would require that the total area under f from 0 to infinity be 1 (i.e., all of the components will eventually fail). In this lab we shall see how to make mathematical sense of these notions.

[1]We shall look more closely at the ideas underlying this application later in the semester.

Part I: The basic ideas

1. Compute by hand each of the following integrals and use your calculator to approximate the answer to six decimal places:

$$\int_1^2 e^{-t}dt \qquad \int_1^{10} e^{-t}dt \qquad \int_1^{100} e^{-t}dt \qquad \int_1^{1000} e^{-t}dt$$

What do you think the value of $\int_1^{\infty} e^{-t}dt$ ought to be? Why?

2. Compute by hand each of the following integrals and use your calculator to approximate the answer to three decimal places. (*Hint: what inverse trigonometric function has a derivative which is* $\frac{1}{1+x^2}$?)

$$\int_0^2 \frac{2}{1+x^2}\,dx \qquad \int_0^b \frac{2}{1+x^2}\,dx$$ (*Hint: your answer will involve the constant b*)

Use your answer to the last step to compute by hand the following limit:

$$\lim_{b\to\infty} \int_0^b \frac{2}{1+x^2}\,dx$$

What do you think the value of $\int_0^{\infty} \frac{2}{1+x^2}\,dx$ should be? Why?

Definition: Integrals of the form $\int_a^{\infty} f(x)dx$ are defined to mean $\lim_{b\to\infty} \int_a^b f(x)dx$. If that limit exists we say that $\int_a^{\infty} f(x)dx$ converges, and if that limit does not exist, we say that $\int_a^{\infty} f(x)dx$ diverges.

Thus, in step 1 above we saw that the integral $\int_1^{\infty} e^{-t}dt$ converges. In other words when we say the integral of a positive function from 1 to infinity converges, we mean that the total area between the function and the horizontal axis from 1 to infinity is a finite number.

Part II: Some computations

1. Consider $\int_1^\infty \frac{1}{t}\,dt$. Does this integral converge? Show why or why not.

2. Does the integral $\int_1^\infty \frac{1}{t^2}\,dt$ converge? Show why or why not.

3. For which values of the constant p does the integral $\int_1^\infty \frac{1}{t^p}\,dt$ converge? Explain.

(*Hint: you will need to consider three cases.*)

Part III: Extending these ideas

1. Graph $y = e^{-t}$ and $y = e^{-t^2}$ on the same set of axes for $t \geq 1$. Use an argument with areas to determine whether $\int_1^\infty e^{-t^2} dt$ converges. Explain your work. Could you answer this question by finding the antiderivative of e^{-t^2}, then evaluating the integral as we did in Part I? Why or why not?

2. Graph $y = \frac{1}{ln\, x}$ and $y = \frac{1}{x}$ on the same set of axes using $[2, \infty]$ as the common domain. Use that graph to decide whether $\int_2^\infty \frac{1}{ln\, x}\,dx$ converges. Explain your work.

3. Assume that $0 \leq f(t) \leq g(t)$ for $t \geq a$. Does convergence of either of the integrals $\int_a^\infty f(t)\,dt$ and $\int_a^\infty g(\mathrm{t})\,dt$ always imply convergence of the other? Explain your answer for both cases.

Part IV: A connection to series

Draw a graph of $\frac{1}{x^3}$ from 1 to ∞. Is the area under the graph to the right of 1 finite or infinite? Now, on the same graph draw a picture that represents $\sum_{k=2}^{\infty} \frac{1}{k^3}$ as an area. (Hint: think about right-hand sums). What can you conclude about the convergence of $\sum_{k=2}^{\infty} \frac{1}{k^3}$? Explain your conclusions.

Part V: Conclusion

In the preview we raised a question about the area under the function $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$ and for some positive constant λ. (That is, can total area under a curve from 0 to infinity be finite?) Recall that the area under the graph of this function over an interval [a,b] is the fraction of certain electrical components which we expect to fail between time a and time b.

1. Compute the total area under f for $t \geq 0$. Why does the answer you get make sense in terms of probability and the context of the problem?

2. Let T be a fixed value of t. $[T > 0]$ Compute each of the following integrals.

$$\int_T^{\infty} \lambda e^{-\lambda t}\, dt \qquad \int_0^T \lambda e^{-\lambda t}\, dt$$

3. Explain in terms of electrical components, time, and percentages what each answer above represents. Also explain their relationship to each other.

Report

Your report should include your responses to each question in this lab, including the graphs.