Varying Density

Purpose

The purpose of this lab is to show how a quantity whose density varies, such as population, can be calculated using integration.

Preview: The strategy we will employ for finding a population whose density varies follows the steps listed below.

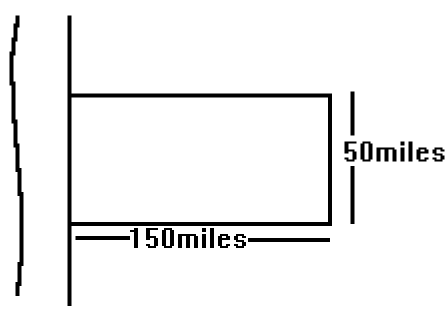
- Divide the region where the population exists into small pieces so that on each piece the population density is approximately constant.
- Estimate the population on each piece assuming the density is constant on that piece.
- Approximate the total population by adding up the contributions to the population from each piece.
- Turn the sum from the last step into a definite integral by taking a limit as the number of subdivisions of the region increases to infinity and their size shrinks to 0.
- Evaluate the definite integral.

Part 1

Pictured below are the borders of a wildlife park . You can see that it is in the shape of a rectangle with a river flowing along one of its boundaries. By inspecting mosquito traps set up about the park, an entomologist has formulated the following model of mosquito concentration :

$$m(x) = \frac{8{,}000{,}000}{3x + 120},$$

where x is the distance from the river measured in miles and $m(x)$ is the mosquito concentration measured in mosquitoes per square mile.



1. Where in the park are the mosquitoes most concentrated? Least concentrated?

2. What is the range of the mosquito concentration in the park? Be sure to include units in your answer.

3. Divide the park into five rectangles of equal width, parallel to the river. Using the midline of each rectangle as its approximate distance from the river, estimate the number of mosquitoes on each rectangle and for the entire park. Include units throughout your calculations.

4. Why would it have been a mistake to divide the park into rectangles running perpendicular to the river?

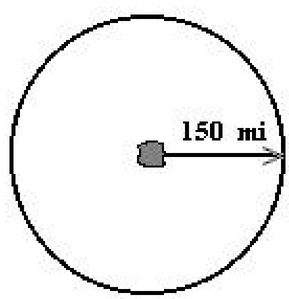
5. How could we improve the estimation of the park's mosquito population made in question 3?

6. Suppose you divide the park into *n* rectangles of equal width, $\Delta x = \frac{150}{n}$, and we let $\widehat{x}_k$ be the distance from the midline of the k-th rectangle to the river. (You do NOT need to find a formula for $\widehat{x}_k$ to complete this step.) Write down a Riemann sum which yields an estimate of the total number of mosquitoes in the park based on these *n* rectangles. Write your answer using summation notation. Clearly explain your work, and make sure your summation yields mosquitoes and not mosquitoes per square mile.

7. Set up and evaluate an integral which will yield the total number of mosquitoes in the park. Use the Fundamental Theorem of Calculus to evaluate the integral. Show all steps of your computations clearly. (Answer: 207752616 mosquitoes.)

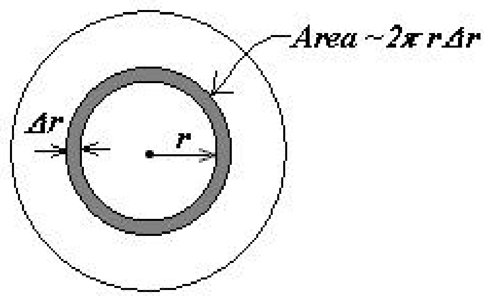
Part 2

For this problem we will assume that the wildlife park is in the shape of a circle with a radius of 150 miles. There is a small pond at the center of the park. We will use the same density function , $m(r) = \frac{8{,}000{,}000}{3r+120}$, we did in part 1, but this time *r* represents the distance from the center of the park.



1. How should we divide the area of the circular park so that the population density of mosquitoes is approximately constant on each piece? For example, slicing the park into wedge shaped pieces like a pizza would be a bad idea. (Why?) Draw a picture showing how you would divide the park.

2. Explain why the area of the shaded piece of the park pictured below can be approximated by $2\pi r\Delta r$, if we assume that Δr is small. Why is $2\pi r\Delta r$ not an exact measure of the area of the shaded ring?



3. Divide the circular park into five concentric rings of equal width. Approximate the number of mosquitoes on each ring and for the park. Your approximation may be an over or under estimate depending upon your choice of r for each ring. The table below may help you organize your work.

	ring 1	ring 2	ring 3	ring 4	ring 5
approximate area in mi^2 ($2\pi r\Delta r$)					
approximate number of mosquitoes per mi^2					
approximate number of mosquitoes					

4. Write down a Riemann sum, with *n* subdivisions, which approximates the mosquito population on the circular park.

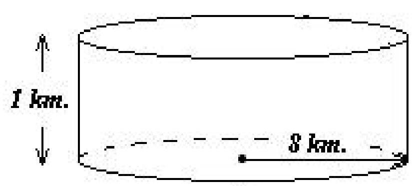
5. Write down a definite integral which yields the mosquito population of the circular park. Evaluate your integral using the numerical integration feature on your calculator. (Answer: 1,468,995,577 mosquitoes)

Part 3

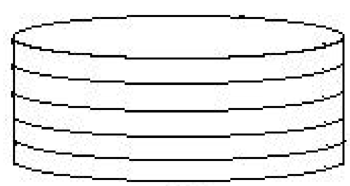
In this problem you will calculate the amount of pollution in the air over a city. Suppose that environmental engineers have analyzed the air over a city at various heights above the ground and have recorded the results of their study in the table below. Pay careful attention to the units given in the table.

height above the ground in meters	0	200	400	600	800	1000
density of pollutant in kg/m^3	0.00040	0.00024	0.00015	0.000089	0.000054	0.000033

Picture the air above this city as a circular cylinder with a radius of 8 kilometers and a height of 1 kilometer as shown below in figure A.



A



B

1. Since the density of the pollutant varies with altitude, we begin by dividing the cylinder of air horizontally into five equal slices as shown in figure B above.

   (a) Find the volume (in m^3) of a single slice of air depicted in figure B. (Recall that the volume of a cylinder is given by $V = \pi r^2 h$)

   (b) Use the information provided in the table to find a lower and an upper estimate for the amount of pollutant (in *kg*) in the cylinder of air above the city. (Answer: upper estimate is 3.75×10^7 *kg*)

2. The exponential function, $p(h) = 0.0004e^{-0.0025h}$, provides an excellent model for the data given in the table, where is height above the ground in meters and $p(h)$ is the density of the pollutant in $\frac{kg}{m^3}$ at an altitude of h.

(a) Using the function $p(h)$ write a Riemann sum which approximates the amount of pollutant in the cylindrical column of air, assuming we have divided the cylinder into *n* slices of equal thickness.

(b) Write down an integral which yields the total amount of pollutant in the air over the city to an altitude of 1000 meters. Use the Fundamental Theorem of Calculus to evaluate the integral. (Answer: 29,529,241 *kg*)

(c) The total amount of pollutant over the city reaches a limit as the height of the cylinder of air increases. Find this limit.

Report

Your report should include your responses to all the questions. Clearly show all of your calculations.