2019 DUKE MATH MEET TEAM ROUND

Problem 1. Zion, RJ, Cam, and Tre decide to start learning languages. The four most popular languages that Duke offers are Spanish, French, Latin, and Korean. If each friend wants to learn exactly three of these four languages, how many ways can they pick courses such that they all attend at least one course together?

Problem 2. Suppose we wrote the integers between 0001 and 2019 on a blackboard as such:

$$000100020003\cdots 20182019.$$

How many 0's did we write?

Problem 3. Duke's basketball team has made x three-pointers, y two-pointers, and z one-point free throws, where x, y, z are whole numbers. Given that $3|x$, $5|y$, and $7|z$, find the greatest number of points that Duke's basketball team could not have scored.

Problem 4. Find the minimum value of $x^2 + 2xy + 3y^2 + 4x + 8y + 12$, given that x and y are real numbers. Note: calculus is **not** required to solve this problem.

Problem 5. Circles C_1, C_2 have radii 1, 2 and are centered at O_1, O_2, respectively. They intersect at points A and B, and convex quadrilateral O_1AO_2B is cyclic. Find the length of AB. Express your answer as $\frac{x}{\sqrt{y}}$, where x, y are integers and y is square-free.

Problem 6. An infinite geometric sequence $\{a_n\}$ has sum

$$\sum_{n=0}^{\infty} a_n = 3.$$

Compute the maximum possible value of the sum

$$\sum_{n=0}^{\infty} a_{3n}.$$

Problem 7. Let there be a sequence of numbers $x_1, x_2, x_3, \cdots$ such that for all i,

$$x_i = \frac{49}{7^{\frac{i}{1010}} + 49}.$$

Find the largest value of n such that

$$\left\lfloor \sum_{i=1}^{n} x_i \right\rfloor \leq 2019.$$

Problem 8. Let X be a 9-digit integer that includes all the digits 1 through 9 exactly once, such that any 2-digit number formed from adjacent digits of X is divisible by 7 or 13. Find all possible values of X.

Problem 9. Two 2025-digit numbers, $428\underbrace{99\ldots99}_{\text{2019 9's}}571$ and $571\underbrace{99\ldots99}_{\text{2019 9's}}428$, form the legs of a right triangle. Find the sum of the digits in the hypotenuse.

Problem 10. Suppose that the side lengths of $\triangle ABC$ are positive integers and the perimeter of the triangle is 35. Let G the centroid and I be the incenter of the triangle. Given that $\angle GIC = 90^\circ$, what is the length of AB?

TEAM ROUND ANSWERS

Team ______________________

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