2019 Duke Math Meet

Problems and Solutions

Saturday 2nd November, 2019

1 Individual Problems

Problem 1.1. Compute the value of N, where

$$N = 818^3 - 6 \cdot 818^2 \cdot 209 + 12 \cdot 818 \cdot 209^2 - 8 \cdot 209^3.$$

Solution. $\boxed{64,000,000.}$

Note that N is simply the binomial expansion of

$$(818 - 2 \cdot 209)^3,$$

so

$$\begin{aligned} N &= (818 - 2 \cdot 209)^3 \\ &= 400^3 = 64,000,000. \end{aligned}$$

□

Problem 1.2. Suppose $x \leq 2019$ is a positive integer that is divisible by 2 and 5, but not 3. If 7 is one of the digits in x, how many possible values of x are there?

Solution. $\boxed{27.}$

If x is divisible by 2 and 5, it must be divisible by 10, thus the last digit must be 0, so there are only 3 variable digits of x. We now do casework on the number of digits of x.

Case 1: 2 digits. x must include the digit 7, so $x = 70$, which is not divisible by 3. Thus, there is 1 2-digit possibilities for x.

Case 2: 3 digits. Since x must include the digit 7, we either have $7a0$ or $a70$, and since x is not divisible by 3, we have a total of $7 + 6 - 1 = 12$ 3-digit possibilities for x.

Case 3: 4 digits. Similar to the previous case, we either have $x = 7ab0$, $a7b0$, or $ab70$. But, since $x \leq 2019$, we must have $x = 1a70$ or $17a0$. Thus, we get a total of $7 + 7 = 14$ 4-digit possibilities for x.

Summing these gives $1 + 12 + 14 = 27$. □

Problem 1.3. Find all non-negative integer solutions (a, b) to the equation

$$b^2 + b + 1 = a^2.$$

Solution. $\boxed{(1, 0).}$

We have

$$b^2 < b^2 + b + 1 = a^2 \leq b^2 + 2b + 1 = (b + 1)^2.$$

Hence, the only way for a to be an integer is if $a = b + 1$. This implies that $(a, b) = (1, 0)$ is the only solution. $\square$

Problem 1.4. Compute the remainder when $\sum_{n=1}^{2019} n^4$ is divided by 53.

Solution. $\boxed{25.}$

Notice that $\sum_{x=1}^{n} x^4$ is equivalent to counting the number of 5-tuples $(x_1, x_2, x_3, x_4, x_5)$ from $\{1, 2, \ldots, n + 1\}$ with $x_5 > \max(x_1, x_2, x_3, x_4)$ (you can see this by doing casework on x_5). The number of 5-tuples is also equal to

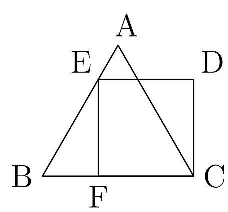
$$\binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5}.$$

This shows that $\sum_{x=1}^{n} x^4$ is divisible by n. Thus, since $2019 = 53 \cdot 38 + 5$, we can reduce our desired sum to

$$1^4 + 2^4 + 3^4 + 4^4 + 5^5 \mod 5,$$

which we can easily compute to be 25. $\square$

Problem 1.5. Let ABC be an equilateral triangle and $CDEF$ a square such that E lies on segment AB and F on segment BC. If the perimeter of the square is equal to 4, what is the area of triangle ABC?



Solution. $\boxed{\frac{1}{2} + \frac{\sqrt{3}}{3}.}$

We easily find $EF = 1$, so since $\angle B = 60°$, we know $BF = \frac{1}{\sqrt{3}}$, so $BC = 1 + \frac{1}{\sqrt{3}}$. Therefore, the area of ABC is

$$\frac{\sqrt{3}}{4}\left(1 + \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{2} + \frac{\sqrt{3}}{3}.$$

$\square$

Problem 1.6.

$$S = \frac{4}{1 \times 2 \times 3} + \frac{5}{2 \times 3 \times 4} + \frac{6}{3 \times 4 \times 5} + \cdots + \frac{101}{98 \times 99 \times 100},$$

Let $T = \frac{5}{4} - S$. If $T = \frac{m}{n}$, where m and n are relatively prime integers, find the value of $m + n$.

Solution. $\boxed{6667.}$

$$\begin{aligned}
S &= \frac{4}{1 \times 2 \times 3} + \frac{5}{2 \times 3 \times 4} + \frac{6}{3 \times 4 \times 5} + ... + \frac{101}{98 \times 99 \times 100} \\
&= (\frac{3}{1 \times 2 \times 3} + \frac{4}{2 \times 3 \times 4} + \frac{5}{3 \times 4 \times 5} + ... + \frac{100}{98 \times 99 \times 100}) \\
&\quad + (\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + ... + \frac{1}{98 \times 99 \times 100}) \\
&= (\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + ... + \frac{1}{98 \times 99}) \\
&\quad + (\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + ... + \frac{1}{98 \times 99 \times 100}) \\
&= (\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{98} - \frac{1}{99}) \\
&\quad + \frac{1}{2}(\frac{1}{1 \times 2} - \frac{1}{2 \times 3} + \frac{1}{2 \times 3} - \frac{1}{3 \times 4} + \ldots + \frac{1}{98 \times 99} - \frac{1}{99 \times 100}) \\
&= (1 - \frac{1}{99}) + \frac{1}{2}(\frac{1}{2} - \frac{1}{9900}) \\
&= \frac{5}{4} - (\frac{1}{99} + \frac{1}{19800}) \\
&= \frac{5}{4} - \frac{67}{6600}.
\end{aligned}$$

Thus, $\frac{m}{n} = \frac{67}{6600}$, giving the answer of 6667. □

Problem 1.7. Find the sum of

$$\sum_{i=0}^{2019} \frac{2^i}{2^i + 2^{2019-i}}.$$

Solution. $\boxed{1010.}$

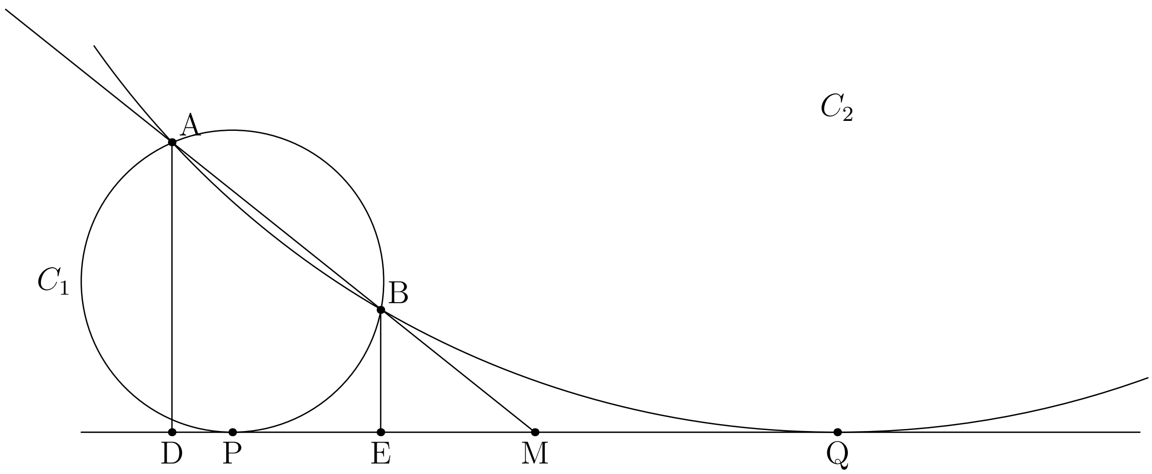
Note that

$$\frac{2^i}{2^i + 2^{2019-i}} + \frac{2^{2019-i}}{2^i + 2^{2019-i}} = 1,$$

so we can pair all the elements in the sum to get 1010. □

Problem 1.8. Let A and B be two points in the Cartesian plane such that A lies on the line $y = 12$, and B lies on the line $y = 3$. Let C_1, C_2 be two distinct circles that intersect both A and B and are tangent to the x-axis at P and Q, respectively. If $PQ = 420$, determine the length of AB.

Solution. $\boxed{315.}$



Draw the line through A and B, and let it hit the x-axis at M. The power of M with respect to C_1 is $PM^2 = BM \cdot AM$, and the power of M with respect to C_2 is $MQ^2 = MB \cdot MA$, so we find that $PM = MQ = 210$. Let D be the perpendicular from A to the x-axis and let E be the perpendicular from B to the x-axis. Then, similar triangles MBE and MAD give us $\frac{BM}{AM} = \frac{3}{12} = \frac{1}{4}$, so combining this with $BM \cdot AM = 210^2$ gives $BM = 105$, so $AM = 420$, and $AB = 315$. □

Problem 1.9. Zion has an average 2 out of 3 hit rate for 2-pointers and 1 out of 3 hit rate for 3-pointers. In a recent basketball match, Zion scored 18 points without missing a shot, and all the points came from 2 or 3-pointers. What is the probability that all his shots were 3-pointers?

Solution. $\boxed{\frac{27}{8435}.}$

The conditional probability tells us that the probability is the probability of shooting all 3-pointers over the probability of scoring 18 points without missing, which is

$$\frac{\left(\frac{1}{3}\right)^6}{\left(\frac{1}{3}\right)^6 + \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^3 \binom{7}{4} + \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^6 \binom{8}{2} + \left(\frac{2}{3}\right)^9} = \frac{27}{8435}.$$

□

Problem 1.10. Let $S = \{1, 2, 3, \ldots, 2019\}$. Find the number of non-constant functions $f : S \to S$ such that

$$f(k) = f(f(k+1)) \le f(k+1) \quad \text{for all } 1 \le k \le 2018.$$

Express your answer in the form $\binom{m}{n}$, where m and n are integers.

Solution. $\boxed{\dbinom{2019}{3}.}$

Suppose $f(1) = a$. Then, $f(f(2)) = a \le f(2)$. If $f(2) > a$, then $f(f(2)) \ge f(a) \ge f(2) > a$, which is a contradiction by the monotonicity of f. Therefore, $f(2) = a$. By a similar argument, we see that $f(a+1) = a$. Now, suppose $f(i) = b$, with $f(j) = a$ with $a < b$ for all $j < i$. Then, $f(f(i+1)) = b \le f(i+1)$. If $f(i+1) = b$, then $f(f(i+1)) \ge f(b) = a$ because $b < i$, which makes a contradiction. On the other hand, if $f(i+1) > b$, then there is no other value $x \in S$ such that $f(x) = b$, so $x = i$, hence $f(i+1) = i$. Therefore, we see that the number of non-constant functions satisfying the given condition is the same as the number of ways to pick a, b, and i, where $a < b < i$. This is simply the number of ways to pick 3 distinct numbers from S, or $\binom{2019}{3}$. □

2 Tiebreakers

Problem 2.1. Let $a(1), a(2), \ldots, a(n), \ldots$ be an increasing sequence of positive integers satisfying $a(a(n)) = 3n$ for every positive integer n. Compute $a(2019)$.

Solution. $\boxed{3870.}$

If $a(1) = 1$ we also have $a(a(1)) = 1 \ne 3 \cdot 1$ which is impossible. Since the sequence is increasing, it follows that $1 < a(1) < a(a(1)) = 3$ and thus $a(1) = 2$. From the equation we deduce $a(3n) = a(a(a(n))) = 3a(n)$ for all n. We easily prove by induction (starting with $a(1) = 2$) that $a(3^m) = 2 \cdot 3^m$ for every m. Using this we also obtain $a(2 \cdot 3^m) = a(a(3^m)) = 3^{m+1}$.

There are $3^n - 1$ integers i such that $3^n < i < 2 \cdot 3^n$ and there are $3^n - 1$ integers j such that $a(3^n) = 2 \cdot 3^n < j < 3^{n+1} = a(2 \cdot 3^n)$. Since $a(n)$ is increasing there is no other option than $a(3^n + b) = 2 \cdot 3^n + b$ for all $0 < b < 3^n$. Therefore $a(2 \cdot 3^n + b) = a(a(3^n + b)) = 3^{n+1} + 3b$ for all $0 < b < 3^n$. Since $2019 = 2 \cdot 3^6 + 561$, we have $a(2019) = 37 + 3 \cdot 561 = 3870$. □

Problem 2.2. Consider the function $f(12x - 7) = 18x^3 - 5x + 1$. Then, $f(x)$ can be expressed as $f(x) = ax^3 + bx^2 + cx + d$, for some real numbers a, b, c and d. Find the value of $(a + c)(b + d)$.

Solution. $\boxed{\frac{135}{64}.}$

The problem asks for a product of some sums of coefficients of $f(x)$, suggesting that values like $f(1)$ and $f(-1)$ are useful in finding the target. We know that $f(1) = a+b+c+d$ and $f(-1) = -a+b-c+d$, so we have that $f(1)+f(-1) = 2(b+d)$ and $f(1)-f(-1) = 2(a+c)$, meaning that

$$(f(1) + f(-1))(f(1) - f(-1)) = 4(a + c)(b + d) = f^2(1) - f^2(-1),$$

by difference of squares. But we know from the given equation that $f(1)$ is obtained when $x = \frac{2}{3}$ and $f(-1)$ when $x = \frac{1}{2}$, so plugging in, we have that

$$f(1) = 18\frac{2}{3}^3 - 5\frac{2}{3} + 1 = 3$$

and

$$f(-1) = 18\frac{1}{2}^3 - 5\frac{1}{2} + 1 = \frac{3}{4}.$$

Therefore, $f^2(1) - f^2(-1) = 9 - \frac{9}{16} = \frac{135}{16}$, so $(a + c)(b + d) = \frac{135}{64}$. □

Problem 2.3. Let a, b be real numbers such that $\sqrt{5 + 2\sqrt{6}} = \sqrt{a} + \sqrt{b}$. Find the largest value of the quantity

$$X = \frac{1}{a + \frac{1}{b + \frac{1}{a + \cdots}}}.$$

Solution. $\boxed{\frac{-3 + \sqrt{15}}{2}.}$

We can easily find $\sqrt{5 + 2\sqrt{6}} = \sqrt{2} + \sqrt{3}$. So, either $a = 2$ and $b = 3$ or $a = 3$ and $b = 2$. Note also that

$$X = \frac{1}{a + \frac{1}{b+X}}.$$

In order to maximize X, we must minimize the denominator, or $a + \frac{1}{b+X}$. This is obviously minimized with $a = 2$, so the expression becomes

$$X = \frac{1}{2 + \frac{1}{3+X}},$$

which we can solve to find $X = \frac{-3+\sqrt{15}}{2}$. □

3 Team Problems

Problem 3.1. Zion, RJ, Cam, and Tre decide to start learning languages. The four most popular languages that Duke offers are Spanish, French, Latin, and Korean. If each friend wants to learn exactly three of these four languages, how many ways can they pick courses such that they all attend at least one course together?

Solution. $\boxed{232.}$

The complement is that they do not have a class together, which happens in $4! = 24$ ways. Since there are $4^4 = 256$ total combinations of classes they can take, the answer is $256 - 24 = 232$. □

Problem 3.2. Suppose we wrote the integers between 0001 and 2019 on a blackboard as such:

$$000100020003 \cdots 20182019.$$

How many 0's did we write?

Solution. $\boxed{1628}$

We do casework on the position of the 0 for each group of 4 digits. Note that we don't have to care about overcounting (if we were counting the group 0001, then we would count it three times, but there are 3 zeros). Consider the integer $0abc$. There are 10 choices for each of a, b, and c, giving us $10 \cdot 10 \cdot 10 = 1000$ total. For the integer $a0bc$, we have $2 \cdot 10 \cdot 10 + 1 \cdot 2 \cdot 10 = 220$ total integers. Similarly, we have $2 \cdot 10 \cdot 10 + 1 \cdot 1 \cdot 10 = 210$ for the integer $ab0c$ and $2 \cdot 10 \cdot 10 + 1 \cdot 1 \cdot 2 = 202$ for the integer $abc0$. However, we are counting 0000 each time, so we must subtract 4, giving us a total of

$$1000 + 220 + 210 + 202 - 4 = 1628.$$

□

Problem 3.3. Duke's basketball team has made x three-pointers, y two-pointers, and z one-point free throws, where x, y, z are whole numbers. Given that $3|x$, $5|y$, and $7|z$, find the greatest number of points that Duke's basketball team could not have scored.

Solution. $\boxed{23.}$

This question is essentially asking to find the greatest integer I that cannot be expressed as the sum of multiples of 7, 9, and 10, so $I \neq 7k + 9l + 10m$ for whole numbers k, l, m. We can find the upper bound of I by setting $m = 0$. $I \equiv 9l \pmod 7$, so any number smaller than $9n$ for $n \leq 6$ that has the same remainder modulo 7 cannot be expressed as $7k + 9l$. The upper bound is therefore $9 \cdot 6 - 7 = 47$. However, $47 = 3 \cdot 9 + 2 \cdot 10$, so $I < 47$. We can find other numbers that cannot be expressed as $7k + 9l$ by deducting multiples of 7 and/or multiples of 9 from 47 and see whether they can be expressed as $7k + 9l + 10m$. It can then be found that 23 is the greatest value. We can check that 24 through 30 can be expressed as $7k + 9l + 10m$.

□

Problem 3.4. Find the minimum value of $x^2 + 2xy + 3y^2 + 4x + 8y + 12$, given that x and y are real numbers. Note: calculus is **not** required to solve this problem.

Solution. $\boxed{6.}$

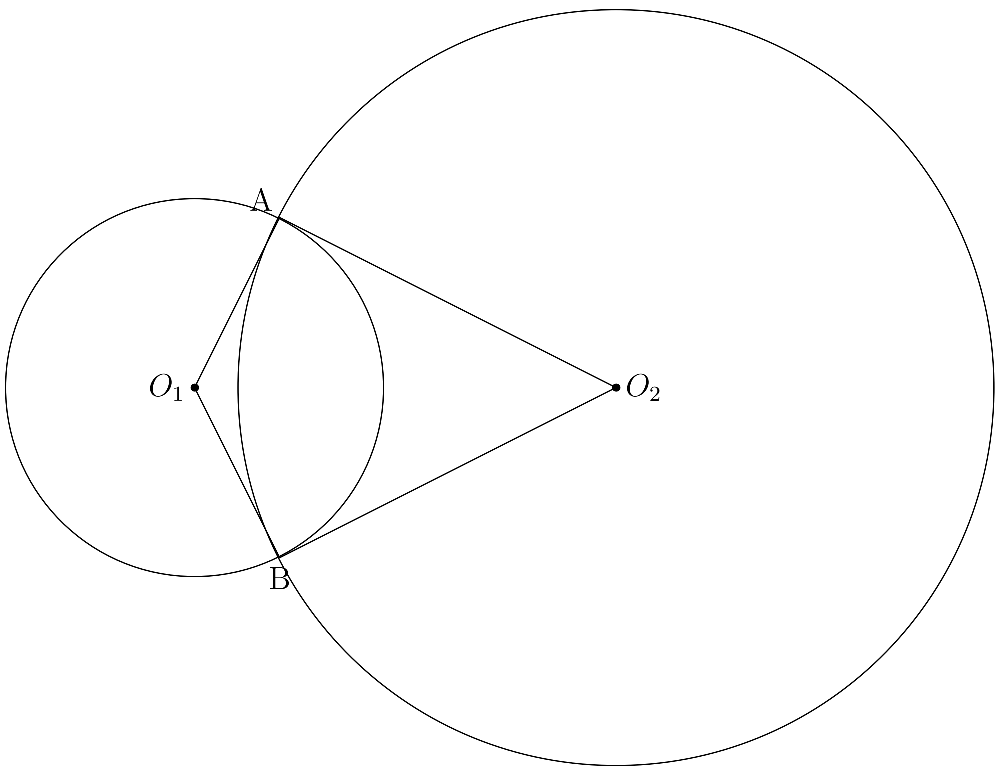
We see that

$$\begin{aligned} x^2 + 2xy + 3y^2 + 4x + 8y + 12 &= x^2 + (2y+4)x + 3y^2 + 8y + 12 \\ &= (x+y+2)^2 + 2y^2 + 4y + 8 \\ &= (x+y+2)^2 + 2(y+1)^2 + 6 \geq 6. \end{aligned}$$

We can check that the expression is equal to 6 when $x = y = -1$, so 6 is attainable, and thus is the minimum value. □

Problem 3.5. Circles C_1, C_2 have radii 1, 2 and are centered at O_1, O_2, respectively. They intersect at points A and B, and convex quadrilateral O_1AO_2B is cyclic. Find the length of AB. Express your answer as $\frac{x}{\sqrt{y}}$, where x, y are integers and y is square-free.

Solution. $\boxed{\frac{4}{\sqrt{5}}.}$



In a cyclic quadrilateral, opposite angles add to $180°$. Since $\angle O_1AO_2 = \angle O_1BO_2$ due to symmetry, they must each be $90°$. Therefore, $O_1O_2 = \sqrt{1^2 + 2^2} = \sqrt{5}$. The area of O_1AO_2B is equal to $2 \times \frac{1 \cdot}{2} = 2$. Equating the area to $\frac{O_1O_2 \cdot AB}{2}$, we get $AB = \frac{4}{\sqrt{5}}$. □

Problem 3.6. An infinite geometric sequence $\{a_n\}$ has sum

$$\sum_{n=0}^{\infty} a_n = 3.$$

Compute the maximum possible value of the sum

$$\sum_{n=0}^{\infty} a_{3n}.$$

Solution. $\boxed{4.}$

We are given that $\frac{a_0}{1-r} = 3$, where r is the ratio between terms in the sequence by the infinite geometric sum. The sum we are asked to calculate is then $\frac{a_0}{1-r^3}$. Dividing these two equations gives us $\frac{1}{1+r+r^2} = \frac{k}{3}$, so $k = \frac{3}{1+r+r^2}$ for $r \in (-1, 1)$. To maximize k, we must minimize the denominator, which is done at $r = -\frac{1}{2}$, giving us $k = 4$. ☐

Problem 3.7. Let there be a sequence of numbers $x_1, x_2, x_3, \cdots$ such that for all i,

$$x_i = \frac{49}{7^{\frac{i}{1010}} + 49}.$$

Find the largest value of n such that

$$\left\lfloor \sum_{i=1}^{n} x_i \right\rfloor \leq 2019.$$

Solution. $\boxed{4039.}$

Note that if $f(x) = \frac{49}{7^x+49}$, then $f(x) + f(4-x) = 1$, which can be seen by simplifying $1 - f(x)$. Then, the sequence up to x_{4039} has 2019 pairs that sum up to 1, the pairs being x_i and $x_{4040=i}$, and one term equal to $\frac{1}{2}$, so the answer is 4039. ☐

Problem 3.8. Let X be a 9-digit integer that includes all the digits 1 through 9 exactly once, such that any 2-digit number formed from adjacent digits of X is divisible by 7 or 13. Find all possible values of X.

Solution. $\boxed{784913526.}$

The only possible 2-digit numbers are

$$13, 14, 21, 26, 28, 35, 39, 42, 49, 52, 56, 63, 65, 77, 78, 84, 91, 98.$$

The strategy is to guess-and-check; if x is the only digit that can precede y, then x must precede y or y is the first digit, and if x is the only digit that can come after y, then x must come after y or y is the last digit. Therefore, 8 must come right before 4, or at the very end. From here, we try all the different combinations until we find a possible value of X, or we find that we cannot add more digits to our number. The result is that we find 784913526 to be the only possible value of X. ☐

Problem 3.9. Two 2025-digit numbers, $428\underbrace{99\ldots99}_{\text{2019 9's}}571$ and $571\underbrace{99\ldots99}_{\text{2019 9's}}428$, form the legs of a right triangle. Find the sum of the digits in the hypotenuse.

Solution. $\boxed{18198.}$

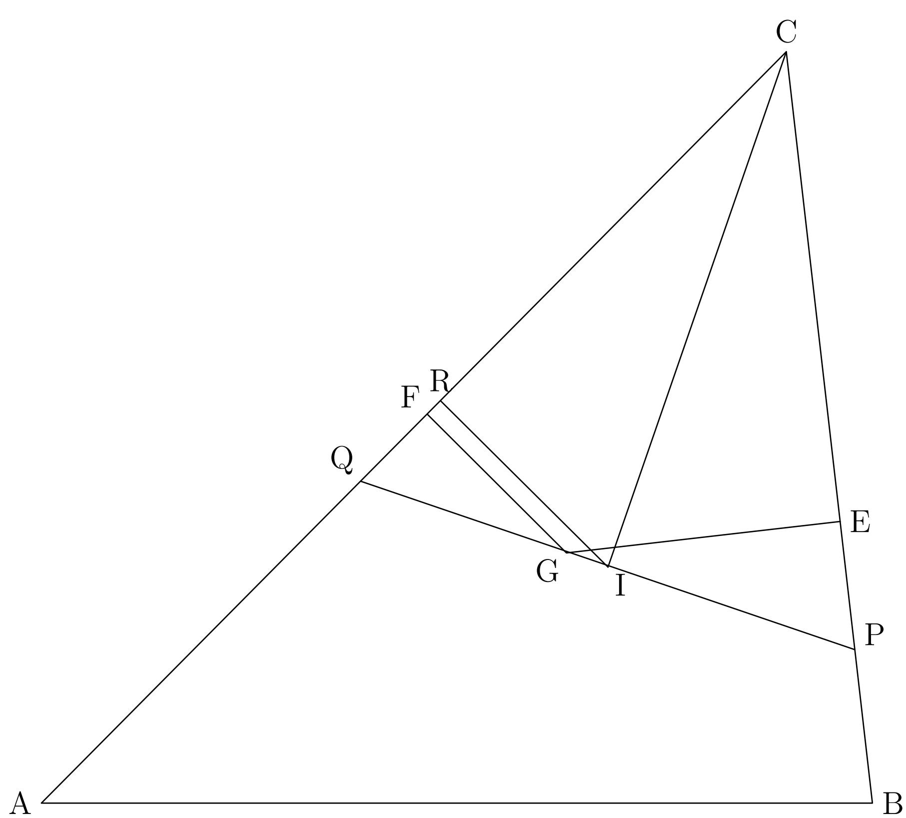
Notice that 428571, 571428, and 714285 form a Pythagorean triple, as seen from repeating parts of $\frac{3}{7}$, $\frac{4}{7}$, and $\frac{5}{7}$. We also notice (by inducting on the number of 9's that we insert) that adding 9's into the middle generates a new Pythagorean triple:

$$4289571^2 + 5719428^2 = 7149285^2$$
$$42899571^2 + 57199428^2 = 71499285^2$$
$$428999571^2 + 571999428^2 = 714999285^2.$$

Thus, the sum of the digits in the hypotenuse is $9 \cdot 2019 + 7 + 1 + 4 + 2 + 8 + 5 = 18198$. Alternatively, one could use the idea that there is no feasible way to find the hypotenuse if there wasn't a pattern to the triple to intelligently guess that the sum of the digits of the hypotenuse should be the same as the sum of the digits of the legs, hence 18198. $\square$

Problem 3.10. Suppose that the side lengths of $\triangle ABC$ are positive integers and the perimeter of the triangle is 35. Let G the centroid and I be the incenter of the triangle. Given that $\angle GIC = 90°$, what is the length of AB?

Solution. $\boxed{11.}$



Let $GI \cap CB = P$, $GI \cap AC = Q$. Let R be the perpendicular from I to QC, F be the perpendicular from G to QC, and E be the perpendicular from G to CP. Then, $IR = r$, the inradius of ABC. Since $PQ \perp CI$ and $\angle QCI = \angle PCI$, we have $PC = QC$ and $PI = QI$. Let $[ABC]$ denote the area of triangle ABC. Then, we have

$$\begin{aligned}[PCQ] &= [GCQ] + [GCP] \\ &= \tfrac{1}{2}CQ(GE + GF) \\ &= \tfrac{1}{2}CQ \cdot IR \cdot 2,\end{aligned}$$

so we get $GE + GF = 2IR = 2r$. Since G is the centroid, if h_a denotes the length of the altitude from A to BC and h_b is defined similarly, we know that $GE = \frac{1}{3}h_a$ and $GF = \frac{1}{3}h_b$. Thus, we substitute to get

$$\frac{1}{3}\left(\frac{2[ABC]}{BC} + \frac{2[ABC]}{AC}\right) = 2\frac{2[ABC]}{a+b+c},$$

so if $BC = a$, $AC = b$, and $AB = c$, then we get

$$6ab = (a+b)(a+b+c) = 35(a+b).$$

Therefore, $6|a+b$, and by Triangle Inequality, $18 \le a + b < 35$, so $a + b = 18$, 24, or 30. Since a and b are both integers, we must have $a + b = 24$, which can be seen by trial and error with the other two cases, which gives us $AB = c = 11$. $\square$

4 Relay Problems

4.1 Problem 1

Problem 4.1.1. We can write 2019 as

$$2019 = a^4 + b^4 + c^4 + d^4 + e^4,$$

where a, b, c, d, and e are integers. Find the minimum value of $|a+b+c+d+e|$, where $|x|$ is the absolute value function.

Solution. $\boxed{1.}$

We can easily find $2019 = 1^4 + 2^4 + 3^4 + 5^4 + 6^4$. There are 3 odd integers and 2 even integers, so we know the minimum value of $|a+b+c+d+e|$ must be odd. We find that $(a, b, c, d, e) = (1, 2, -3, 5, -6)$ gives $|a+b+c+d+e| = 1$, so the minimum value is 1. □

Problem 4.1.2. Let T = TNYWR, and let $ABCDEFGH$ be a regular octagon centered at O with $AO = 4T$. Determine the area of the incircle of $ACEG$.

Solution. $\boxed{8\pi.}$

Before receiving the value of T, notice that the side length of $ACEG$ is $AC = \sqrt{2}AO = 4\sqrt{2}T$, since AOC is a 45-45-90 triangle. Then, the radius of the incircle is $\frac{4\sqrt{2}T}{2} = 2\sqrt{2}T$, so the area of the incircle is $8T^2\pi$. Since $T = 1$, the answer is 8π. □

Problem 4.1.3. Let T = TNYWR, and let $N = \frac{T}{\pi}$. Suppose a bag of 15 apples has N rotten apples. What is the probability that if I randomly pick apples from the bag without replacement, the 11^{th} apple I draw is the last rotten one? Express your answer as a common fraction.

Solution. $\boxed{\dfrac{8}{429}.}$

Before receiving the value of T, notice that the probability that the 11^{th} apple is the last rotten one is equivalent to the the probability of a sequence of 15 letters, N of which are "R" and $15-N$ of which are "C" (for rotten and clean), has the last R on the 11^{th} slot. This probability is equal to number of ways to order 10 letters with $N-1$ labeled "R", divided by the total number of ways to order the 15 letters. Thus, it is equal to

$$\frac{\binom{10}{N-1}}{\binom{15}{N}}.$$

Since $T = 8\pi$, $N = 8$, so our answer is

$$\frac{\binom{10}{7}}{\binom{15}{8}} = \frac{8}{429}.$$

□

4.2 Problem 2

Problem 4.2.1. A cafeteria has 4 entrees and 5 desserts. How many different meals can Jung eat if he eats either 1 or 2 entrees and either 1 or 2 desserts?

Solution. $\boxed{150.}$

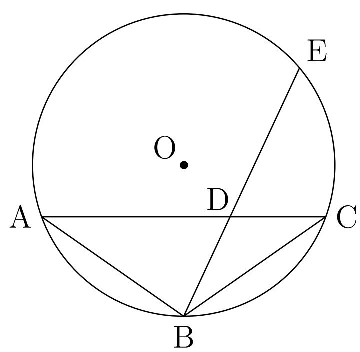
He has $\left(\binom{4}{1}+\binom{4}{2}\right)\left(\binom{5}{1}+\binom{5}{2}\right)=150$ options. □

Problem 4.2.2. Let T = TNYWR. Find the total number of positive integers $n \le T$ such that n^4+5n^2+9 is not divisible by 5.

Solution. $\boxed{30.}$

Before receiving T: notice that $n^4+5n^2+9=(n-1)(n+1)(n^2+1)+5(n^2+2)$, so if $n \equiv \pm 1 \mod 5$, then either $5|(n-1)$ or $5|(n+1)$, and if $n \equiv \pm 2 \mod 5$, then $5|(n^2+1)$. However, if $5|n$, then n^4+5n^2+9 is not divisible by 5, so the answer is the number of multiples of 5 less than or equal ot T. $T=150$, so the answer is $\frac{150}{5}=30$. □

Problem 4.2.3. Let T = TNYWR. In the diagram below, let $AB=BC=\frac{T}{2}$, and $BD=\frac{T}{3}$. Find the length of DE.



Solution. $\boxed{\frac{25}{2}.}$

Let $AD=m$ and $DC=n$. Before receiving the value of T, we apply Stewart's theorem on $\triangle ABC$ to find that

$$\begin{aligned} AD\cdot AC\cdot DC+AC\cdot BD^2 &= DC\cdot AB^2+AD\cdot BC^2 \\ (m+n)\cdot(mn+\tfrac{T^2}{9}) &= \tfrac{T^2}{4}(m+n) \\ mn+\tfrac{T^2}{9} &= \tfrac{T^2}{4} \\ mn &= \tfrac{5T^2}{36}. \end{aligned}$$

By Power of a Point on D, we see that $AD\cdot DC=BD\cdot DE$, so

$$DE=\frac{AD\cdot DC}{BD}=\frac{mn}{T/3}=\frac{5T^2/36}{T/3}=\frac{5T}{12}.$$

Since $T=30$, we get $DE=\frac{150}{12}=\frac{25}{2}$. □