Power Round

1 Dependence and Independence

Definition 1. A *vector* v in $\mathbb{R}^n$ is a collection of real numbers $(v_1, \dots, v_n)$. The sum of two vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ is

$$v + w = (v_1 + w_1, \dots, v_n + w_n).$$

For any constant $c \in \mathbb{R}$, we can multiply a vector v by the constant c to get

$$cv = (cv_1, \dots, cv_n).$$

Definition 2. We say that a collection of vectors $x_1, x_2, \dots, x_k$ is *linearly dependent* if there exist constants $c_1, \dots, c_k$ **not all zero** such that

$$c_1x_1 + c_2x_2 + \dots + c_kx_k = 0.$$

Definition 3. We say that a collection of vectors $x_1, \dots, x_k$ is *linearly independent* if such constants do not exist, i.e. whenever

$$c_1x_1 + c_2x_2 + \dots + c_kx_k = 0,$$

we must have $c_1 = c_2 = \dots = c_k = 0$.

Problem 1.

(a) (2 points) Prove that the vectors $(1, 2, 0)$, $(1, -1, 1)$, and $(3, 0, 2)$ are linearly dependent.

(b) (2 points) Prove that the vectors $(1, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$ are linearly independent.

(c) (2 points) Find a set of n linearly independent vectors $x_1, \dots, x_n \in \mathbb{R}^n$ for every n.

Definition 4. We say that a collection of vectors $x_1, \dots, x_k$ are *affinely dependent* if there exist constants $c_1, \dots, c_n$ **not all zero** such that

$$c_1x_1 + c_2x_2 + \dots + c_kx_k = 0 \qquad \text{and} \qquad c_1 + c_2 + \dots + c_k = 0.$$

Definition 5. We say that a collection of vectors $x_1, \dots, x_k$ are *affinely independent* if such constants do not exist.

Problem 2.

(a) (2 points) Prove that the vectors $(1, 2, 1)$, $(1, -1, 1)$, $(1, 0, 1)$ are affinely dependent.

(b) (2 points) Prove that the vectors $(1, 2, 0)$, $(1, -1, 1)$, and $(3, 0, 2)$ are affinely independent.

Problem 3. (2 points) Prove that if $x_1, \dots, x_k$ are linearly independent, then they are affinely independent.

You may assume the fact that for vectors in $\mathbb{R}^n$, the maximum number of **linearly independent** vectors is n. Therefore if we have $n + 1$ vectors $x_1, \dots, x_{n+1} \in \mathbb{R}^n$ these must be **linearly dependent**.

Problem 4. (2 points) Find an example where equality is reached, i.e. find n linearly independent vectors $x_1, \dots, x_n$ in $\mathbb{R}^n$.

Problem 5.

(a) (2 points) Find $n + 1$ affinely independent vectors in $\mathbb{R}^n$.

(b) (4 points) Prove that the maximum number of affinely independent vectors in $\mathbb{R}^n$ is $n + 1$.

2 Convexity

Definition 6. The *convex hull* of a set of vectors S is the set of all vectors that can be written as

$$c_1 x_1 + \ldots + c_k x_k$$

for some $x_1, \ldots, x_k \in S$, and some $c_1, \ldots, c_k \geq 0$ such that $c_1 + \ldots + c_k = 1$.

Definition 7. A set of vectors S is called *convex* if it is equal to its own convex hull.

Theorem 1 (Radon's Theorem)**.** Let $S = \{x_1, \ldots, x_{n+2}\}$ be any set of $n+2$ points in $\mathbb{R}^n$. Then S can be decomposed into two nonempty disjoint subsets A and B such that the convex hulls of A and B intersect.

Problem 6. (2 points) Verify Radon's Theorem for the points $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ by finding disjoint sets A and B and a point p such that p is in the convex hulls of both A and B.

Problem 7. Prove Radon's Theorem by completing the following steps where notation is as in Theorem 1:

(a) (1 point) Prove that there exist constants $c_1, \ldots, c_{n+2}$, not all zero, such that $\sum_i c_i x_i = 0$ and $\sum_i c_i = 0$.

(b) (2 points) Let I be the set of all indices i such that $c_i > 0$. Show that I is nonempty.

(c) (2 points) Let J be the set of all indices j such that $c_j \leq 0$. Show that

$$\sum_{i \in I} c_i = \sum_{j \in J} -c_j$$

and show that this common sum is not zero.

(d) (4 points) Show that

$$\sum_{i \in I} c_i x_i = \sum_{j \in J} -c_j x_j$$

and use this to show that the convex hulls of $A = \{x_i \ : \ i \in I\}$ and $B = \{x_j \ : \ j \in J\}$ intersect.

Theorem 2 (Helly's Theorem)**.** Let $X_1, \ldots, X_k$ be a collection of convex sets in $\mathbb{R}^n$ with $k > n+1$. If the intersection of any $n+1$ of these is nonempty, the whole collection has a nonempty intersection.

Problem 8 (Proof of Helly's Theorem)**.**

(a) (5 points) Let $k = n+2$ and assume the setup for Helly's Theorem. Let x_j be a point in the intersection of all X_i aside from X_j for $j = 1, \ldots, n+2$ (which exists by the Theorem's assumptions). Use Radon's Theorem to prove Helly's Theorem, i.e. to find a point $p \in \bigcap_i X_i$.

(b) (4 points) Use induction on k to prove Helly's Theorem for $k \geq n+2$ (HINT: Try to collapse two sets by replacing them with their intersection, and show that the hypothesis of Helly's Theorem is still satisfied. You will have to use part (a) both for the base case and the inductive step).

3 Applications of Helly's Theorem

For the following problems you may assume Helly's Theorem as stated above.

Problem 9. (2 points) Given k points in the plane such that every three are contained in a disk of radius 1, prove that all k points are contained in a disk of radius 1 (You may assume that disks are convex).

Problem 10. (2 points) Use the above problem to show that given k points in the plane such that the distance between any two points is at most 1, there is a disk of radius $\frac{1}{\sqrt{3}}$ that contains all k points.

Problem 11. (3 points) Let $S \subset \mathbb{R}^3$ be the unit sphere in 3 dimensions, i.e. the surface of the unit ball, and let $s_1, \ldots, s_k$ be closed hemispheres (i.e. a hemisphere that includes their equator, note that hemispheres need not be parallel to any of the axes). Prove that if

$$\bigcup_{i=1}^{k} s_i = S$$

i.e. that the s_i cover S, then there exist 4 hemispheres $s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4}$ that cover S, i.e.

$$\bigcup_{j=1}^{4} s_{i_j} = S$$

Problem 12 (Generalization of Helly's Theorem)**.** (3 points) For a set of vectors S and a vector p, let $p + S = \{p + s : s \in S\}$ denote the set S translated by the vector p (e.g. if S is a circle centered at the origin, then $p + S$ is a circle centered at p with the same radius). Let $K \subset \mathbb{R}^n$ be a set and let $S = \{X_1, \ldots, X_k\}, X_i \subset \mathbb{R}^n$ be a collection of convex sets with $k > n + 1$. Prove that if the intersection of any $n + 1$ sets of S contains a translated copy of K, i.e. for every $X_{i_1}, \ldots, X_{i_{n+1}}$ there exists a point p such that

$$p + K \subset \bigcap_{j=1}^{n+1} X_{i_j}$$

then the intersection of all k sets contains a translated copy of K, i.e. there exists a point p such that

$$p + K \subset \bigcap_{i=1}^{k} X_i.$$

Note that the original Helly's Theorem can be recovered by letting K be a point.
HINT: You do not need to copy the proof of Helly's Theorem. Try to transform the problem so that you can directly apply Helly's Theorem as stated above.