Power Round

1 Dependence and Independence

Definition 1. A *vector* v in $\mathbb{R}^d$ is a collection of real numbers $(v_1, \ldots, v_n)$. The sum of two vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ is

$$v + w = (v_1 + w_1, \ldots, v_n + w_n).$$

For any constant $c \in \mathbb{R}$, we can multiply a vector v by the constant c to get

$$cv = (cv_1, \ldots, cv_n).$$

Definition 2. We say that a collection of vectors $x_1, x_2, \ldots, x_k$ is *linearly dependent* if there exist constants $c_1, \ldots, c_k$ **not all zero** such that

$$c_1x_1 + c_2x_2 + \ldots + c_kx_k = 0.$$

Definition 3. We say that a collection of vectors $x_1, \ldots, x_k$ is *linearly independent* if such constants do not exist, i.e. whenever

$$c_1x_1 + c_2x_2 + \ldots + c_kx_k = 0,$$

we must have $c_1 = c_2 = \ldots = c_k = 0$.

Problem 1.

(a) (2 points) Prove that the vectors $(1, 2, 0)$, $(1, -1, 1)$, and $(3, 0, 2)$ are linearly dependent.

> **Solution:** We have that
>
> $$1 \cdot (1, 2, 0) + 2 \cdot (1, -1, 1) - 1 \cdot (3, 0, 2) = 0$$
>
> so they are linearly dependent.

(b) (2 points) Prove that the vectors $(1, 0, 0)$, $(0, 1, 1)$, $(1, 0, 1)$ are linearly independent.

> **Solution:** Assume that
>
> $$c_1(1, 0, 0) + c_2(0, 1, 1) + c_3(1, 0, 1) = 0.$$
>
> Then $c_1 + c_3 = 0$, $c_2 = 0$, and $c_2 + c_3 = 0$. From the second and third equations we get that $c_2 = c_3 = 0$ and plugging this into the first equation we get that $c_1 = c_2 = c_3 = 0$. Therefore these vectors are linearly independent.

Definition 4. We say that a collection of vectors $x_1, \ldots, x_k$ are *affinely dependent* if there exist constants $c_1, \ldots, c_n$ **not all zero** such that

$$c_1x_1 + c_2x_2 + \ldots + c_kx_k = 0 \qquad \text{and} \qquad c_1 + c_2 + \ldots + c_k = 0.$$

Definition 5. We say that a collection of vectors $x_1, \ldots, x_k$ are *affinely independent* if such constants do not exist.

Problem 2.

(a) (2 points) Prove that the vectors $(1, 2, 1)$, $(1, -1, 1)$, $(1, 0, 1)$ are affinely dependent.

Solution: Note that
$$1 \cdot (1,2,1) + 2 \cdot (1,-1,1) - 3 \cdot (1,0,1) = 0$$
and $1 + 2 - 3 = 0$ so these are affinely dependent.

(b) (2 points) Prove that the vectors $(1,2,0)$, $(1,-1,1)$, and $(3,0,2)$ are affinely independent.

Solution: Assume that
$$c_1(1,2,0) + c_2(1,-1,1) + c_3(3,0,2) = 0$$
and $c_1 + c_2 + c_3 = 0$. Then we get that $c_1 + c_2 + 3c_3 = 0$, $c_1 - c_2 = 0$ and $c_1 + c + 2 + 2c + 3 = 0$. Substituting $c_1 + c_2 + c_3 = 0$ into either the first or third equations gives that $c_3 = 0$. Therefore $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. Adding these gives $c_1 = 0$ and subtracting gives $c_2 = 0$ so $c_1 = c_2 = c_3 = 0$.

Problem 3. (2 points) Prove that if $x_1, \ldots, x_k$ are linearly independent, then they are affinely independent.

Solution: Assume that $x_1, \ldots, x_k$ are linearly independent. Now assume that we have constants $c_1, \ldots, c_k$ such that
$$c_1x_1 + \ldots + c_kx_k = 0$$
and $c_1 + \ldots + c_k = 0$. Then by linear independence the above equality implies that $c_1 = c_2 = \ldots = c_k = 0$ so these are affinely independent.
(Maybe remove as a problem? Seems kind of trivial/just testing proof writing)

It is known that for vectors in $\mathbb{R}^n$, the maximum number of **linearly independent** vectors is n. Therefore if we have $n+1$ vectors $x_1, \ldots, x_{n+1} \in \mathbb{R}^n$ these must be **linearly dependent**.

Problem 4. (2 points) Find an example where equality is reached, i.e. find n linearly independent vectors $x_1, \ldots, x_n$ in $\mathbb{R}^n$.

Problem 5.

(a) (2 points) Find $n+1$ affinely independent vectors in $\mathbb{R}^n$.

(b) (4 points) Prove that the maximum number of affinely independent vectors in $\mathbb{R}^n$ is $n+1$.

2 Convexity

Definition 6. The *convex hull* of a set of vectors S is the set of all vectors that can be written as
$$c_1x_1 + \ldots + c_kx_k$$
for some $x_1, \ldots, x_k \in S$, and some $c_i \geq 0$ such that $c_1 + \ldots + c_k = 1$.

Definition 7. A set of vectors S is called *convex* if it is equal to its own convex hull.

Theorem 1 (Radon's Theorem)**.** Let $S = \{x_1, \ldots, x_{n+2}\}$ be any set of $n+2$ points in $\mathbb{R}^n$. Then S can be decomposed into two disjoint subsets A and B such that the convex hulls of A and B intersect.

Problem 6. (2 points) Verify Radon's theorem for the points $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ by finding sets A and B and a point p such that p is in the convex hulls of both A and B.

Solution: Let $A = \{(0,0),(1,1)\}$ and $B = \{(0,1),(1,0)\}$. Then the convex hull of A is the segment connecting $(0,0)$ and $(1,1)$ and the convex hull of B is the segment connecting $(0,1)$ and $(1,0)$. These intersect at the point $(1/2, 1/2)$ so we can let $p = (1/2, 1/2)$ and we are done.

Problem 7. Prove Radon's theorem by completing the following steps:

(a) (1 point) Prove that there exist constants $c_1, \ldots, c_{n+2}$, not all zero, such that $\sum_i c_i x_i = 0$ and $\sum_i c_i = 0$.

Solution: By problem 5, $c_1, \ldots, c_{n+2}$ are affinely dependent so such constants must exist.

(b) (2 points) Let I be the set of all indices i such that $c_i > 0$. Show that I is non-empty.

Solution: Assume that no such i exists. Then if there were an i such that $c_i < 0$, the total sum $\sum_i c_i$ would be strictly negative contradicting the assumption that $\sum_i c_i = 0$. Therefore all of the c_i would have to be 0, contradicting that the constants are not all zero. Therefore we have reached a contradiction and such an i must exist so I is nonempty.

(c) (2 points) Let J be the set of all indices j such that $c_j \leq 0$. Show that

$$\sum_{i \in I} c_i = \sum_{j \in J} -c_j$$

and show that this common sum is not zero.

Solution: We can rearrange the identity as

$$\sum_{i \in I} c_i + \sum_{j \in J} c_j = 0$$

which follows because I and J form a partition of $\{1, \ldots, n+2\}$. Because I is nonempty and for all $i \in I$, $c_i > 0$ we must have that the first sum is nonzero, and since the two sums are equal, neither sum can be zero.

(d) (4 points) Show that

$$\sum_{i \in I} c_i x_i = \sum_{j \in J} -c_j x_j$$

and use this to show that the convex hulls of $A = \{x_i \ : \ i \in I\}$ and $B = \{x_j \ : \ j \in J\}$ intersect.

Solution: This again follows from rearranging this as

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j x_j = 0$$

which follows from the definition of the c_i and the fact that I, J form a partition of $\{1, \ldots, n+2\}$. Let the common sum in the previous part be s. Because $s \neq 0$ we can divide this equality by s to get

$$\sum_{i \in I} \frac{c_i}{s} x_i = \sum_{j \in J} \frac{-c_j}{s} x_j$$

Let this common value be p. Then $\sum_{i \in I} \frac{c_i}{s} = 1$ and $\sum_{j \in J} \frac{-c_j}{s} = 1$, and all of these coefficients are non-negative. Therefore p is in the convex hull of both $\{x_i : i \in I\}$ and $\{x_j : j \in J\}$ so we are done.

Theorem 2 (Helly's Theorem)**.** Let $X_1, \ldots, X_k$ be a collection of convex sets in $\mathbb{R}^n$ with $k > n+1$. If the intersection of any $n+1$ of these is non-empty, the whole collection has a nonempty intersection.

Problem 8 (Proof of Helly's Theorem)**.**

(a) (5 points) Let $k = n+2$ and assume the setup for Helly's Theorem. Let x_j be a point in the intersection of all X_i aside from X_j for $j = 1, \ldots, n+2$. (which exists by the theorem's assumptions). Use Radon's Theorem to prove Helly's theorem, i.e. to find a point $p \in \bigcap_i X_i$.

Solution: By Radon's Theorem, there exists a partition I, J of $\{1, \ldots, n+2\}$ and a point p such that p is in the convex hulls of both $\{x_i : i \in I\}$ and $\{x_j : j \in J\}$. I claim that $p \in \bigcap_i X_i$. Fix an i. Then either $i \in I$ or $i \in J$. If $i \in I$, then p is in the convex hull of $\{x_j : j \in J\}$. Note that $x_j \in X_i$ for $j \in J$ because $i \notin J$. Therefore p can be written as a convex combination of points $\{x_j\}_{j \in J}$ each of which is in x_i. Since X_i is convex, this implies that p is an element of X_i. Therefore p is an element of each X_i so p is an element of their intersection, so we are done.

(b) (4 points) Use induction on k to prove Helly's Theorem for $k \geq n+2$ (HINT: Try to collapse two sets by replacing them with their intersection, and show that the hypothesis of Helly's Theorem is still satisfied. You will have to use part (a) both for the base case and the inductive step).

Solution: The base case for $k = n+2$ is proved above, so let us assume that the statement is true for some $k \geq n+2$. Then assume that we have sets $X_1, \ldots, X_{k+1}$ such that the intersection of any $n+1$ is non-empty. Then consider the sets $X_1, \ldots, X_{k-1}, X_k \cap X_{k+1}$. I claim that the intersection of any $n+1$ of these is nonempty. The collections that do not include $X_k \cap X_{k+1}$ automatically satisfy this due to the problem's hypothesis. Therefore it suffices to show that it is true for sets of the form $\{X_{i_1}, \ldots, X_{i_n}, X_k \cap X_{k+1}\}$. Consider the set $\{X_{i_1}, \ldots, X_{i_n}, X_k, X_{k+1}\}$. By Helly's Theorem, this has a nonempty intersection, i.e. there is a point p in their intersection. Then this point p will also lie in the intersection of $\{X_{i_1}, \ldots, X_{i_n}, X_k \cap X_{k+1}\}$, so these sets must have a nonempty intersection. Therefore by the inductive step, there exists a point p in the intersection of $X_1, \ldots, X_{k-1}, X_k \cap X_{k+1}$, and this point p is therefore in each of the X_i, so $p \in \bigcap_i X_i$ as desired. This completes the induction.

3 Applications of Helly's Theorem

For the following problems you may assume Helly's Theorem as stated above.

Problem 9. (2 points) Given k points in the plane such that every three are contained in a disk of radius 1, prove that all k points are contained in a disk of radius 1 (You may assume that disks are convex).

Solution: Let the points be $x_1, \ldots, x_k$ and associate with each point x_i a unit disk D_i centered at x_i. Then the statement that any three are contained in a disk of radius 1 implies that for any three disks, D_i, D_j, D_k their intersection is nonempty. Since disks are convex, Helly's Theorem tells us that the intersection of all of the disks is non-empty. Let p be this common intersection point. Then the distance between p and x_i for any i is at most 1, so all of the x_i lie in a circle of radius 1 centered around p.

Problem 10. (3 points) Use the above problem to show that given k points in the plane such that the distance between any two points is at most 1, there is a disk of radius $\frac{1}{\sqrt{3}}$ that contains all k.

Solution: Consider three points x_1, x_2, x_3. I claim that their circumradius is at most $\frac{1}{\sqrt{3}}$ which would imply the desired statement since we could simply apply the above problem scaled down by a factor of $\frac{1}{\sqrt{3}}$. First we can scale the triangle x_1, x_2, x_3 up until at least one of the sides is exactly equal to 1. It suffices to show that this new triangle has circumradius at most $\frac{1}{\sqrt{3}}$. Assume that x_2x_3 has length 1. Then fix x_2, x_3 and vary the x_1 around the circumcircle of x_2, x_3. We can continue to move it until x_1 is equidistant from x_2, x_3 and makes x_1, x_2, x_3 an isosceles acute triangle. Since this process doesn't change the circumradius, it suffices to to show that all isosceles acute triangles where the base has length 1 has the desired property. If x_1x_2 and x_2x_3 are less than 1, shift x_1 away from x_2x_3 until x_1, x_2, x_3 is equilateral. Since for an acute triangle the circumcircle is the smallest circle that covers the triangle, this can only increase the circumradius. Therefore it suffices to show that an equilateral triangle with side length 1 has circumradius at most $\frac{1}{\sqrt{3}}$, and this is in fact an equality so we are done.
NOTE: Is there a faster way to show this inequality?

Problem 11. (3 points) Let $S \subset \mathbb{R}^3$ be the unit sphere in 3 dimensions and let $s_1, \ldots, s_k$ be closed hemispheres (i.e. a hemisphere that includes the equator). Prove that if

$$\bigcup_{i=1}^{k} s_i = S$$

i.e. that the s_i cover S, then there exist 4 hemispheres $s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4}$ that cover S, i.e.

$$\bigcup_{j=1}^{4} s_{i_j} = S$$

Solution: Let X_i be the convex hull of $S \setminus s_i$. I claim that for a subset $I \subset \{1, \ldots, k\}$, $\bigcap_{i \in I} X_i$ is non-empty if and only if $\bigcup_{i \in I} s_i$ is not all of S. If

$$\bigcup_{i \in I} s_i \neq S$$

there is a point $p \in S$ such that $p \in S \setminus s_i$ for $i \in I$, so $p \in \bigcap_{i \in I} X_i$. On the other hand, if $p \in \bigcap_{i \in I} X_i$, then if we project p onto the surface of the hemisphere, it will still not be an element of $S \setminus s_i$ for $i \in I$ because X_i is conical. Therefore $p \notin \bigcup_{i \in I} s_i$ so these do not cover S.
Now assume that no 4 hemispheres cover S. Then the intersection of any four of the X_i is non-empty. Therefore by Helly's Theorem, the intersection of all of the X_i is nonempty, so the s_i do not all cover S, contradicting the assumption in the problem statement that they do cover S. Therefore some set of 4 hemispheres must cover S.

Problem 12 (Generalization of Helly's Theorem)**.** (4 points) For a set of vectors S and a vector p, let $p + S = \{p + s : s \in S\}$ denote the set S translated by the vector p (e.g. if S is a circle centered at the origin, then $p + S$ is a circle centered at p with the same radius). Let $K \subset \mathbb{R}^n$ be a set and let $S = \{X_1, \ldots, X_k\}, X_i \subset \mathbb{R}^n$ be a collection of convex sets with $k > n + 1$. Prove that if the intersection of any $n + 1$ sets of S contains a translated copy of K, i.e. for every $X_{i_1}, \ldots, X_{i_{n+1}}$ there exists a point p such that

$$p + K \subset \bigcap_{j=1}^{n+1} X_{i_j}$$

then the intersection of all k sets contains a translated copy of K, i.e. there exists a point p such that

$$p + K \subset \bigcap_{i=1}^{k} X_i.$$

Note that the original Helly's theorem can be recovered by letting K be a point.
HINT: You do not need to copy the proof of Helly's Theorem. Try to transform the problem so that you can directly apply Helly's theorem as stated above.

Solution: Let $Y_i = \{x \in \mathbb{R}^n : x + K \subset X_i\}$. I claim that the Y_i are convex. Let $y_1, \ldots, y_k \in Y_i$ and let $c_1, \ldots, c_k$ be non-negative numbers summing to 1. Then I claim that $c_1 y_1 + \ldots + c_k y_k \in Y_i$. We need to show that

$$c_1 y_1 + \ldots + c_k y_k + K \subset X_i$$

This is equivalent to showing that for every $k \in K$, we have

$$c_1 y_1 + \ldots + c_k y_k + k \in X_i$$

however this is equivalent to

$$c_1(y_1 + k) + \ldots + c_k(y_k + k) \in X_i$$

since $\sum_i c_i = 1$. Since $y_j + k \in X_i$ and X_i is convex, this statement is true.
Now the problem statement is equivalent to the fact that the intersection of any $n + 1$ of the Y_i is nonempty. By Helly's Theorem, this implies that the intersection of all of the Y_i is nonempty. Let a point in this intersection be p. Then $p + K$ is a subset of X_i for every i, so $p + K \subset \bigcap_{i=1}^{k} X_i$.