Problem 1. Compute the value of $N$, where

$$N = 818^3 - 6 \cdot 818^2 \cdot 209 + 12 \cdot 818 \cdot 209^2 - 8 \cdot 209^3.$$ 

Problem 2. Suppose $x \leq 2019$ is a positive integer that is divisible by 2 and 5, but not 3. If 7 is one of the digits in $x$, how many possible values of $x$ are there?
Problem 3. Find all non-negative integer solutions \((a, b)\) to the equation
\[ b^2 + b + 1 = a^2. \]

Problem 4. Compute the remainder when \(\sum_{n=1}^{2019} n^4\) is divided by 53.
Problem 5. Let $ABC$ be an equilateral triangle and $CDEF$ a square such that $E$ lies on segment $AB$ and $F$ on segment $BC$. If the perimeter of the square is equal to 4, what is the area of triangle $ABC$?

Problem 6. \[ S = \frac{4}{1 \times 2 \times 3} + \frac{5}{2 \times 3 \times 4} + \frac{6}{3 \times 4 \times 5} + \cdots + \frac{101}{98 \times 99 \times 100}, \]

Let $T = \frac{5}{4} - S$. If $T = \frac{m}{n}$, where $m$ and $n$ are relatively prime integers, find the value of $m + n$. 

**ANSWER TO PROBLEM 5**  

**ANSWER TO PROBLEM 6**
Problem 7. Find the sum of
\[
\sum_{i=0}^{2019} \frac{2^i}{2^i + 2^{2019-i}}.
\]

Problem 8. Let \(A\) and \(B\) be two points in the Cartesian plane such that \(A\) lies on the line \(y = 12\), and \(B\) lies on the line \(y = 3\). Let \(C_1, C_2\) be two distinct circles that intersect both \(A\) and \(B\) and are tangent to the \(x\)-axis at \(P\) and \(Q\), respectively. If \(PQ = 420\), determine the length of \(AB\).
2019 DUKE MATH MEET INDIVIDUAL ROUND

Problems 9-10

Name __________________

Time Limit: 10 minutes

Team _________________

**Problem 9.** Zion has an average 2 out of 3 hit rate for 2-pointers and 1 out of 3 hit rate for 3-pointers. In a recent basketball match, Zion scored 18 points without missing a shot, and all the points came from 2 or 3-pointers. What is the probability that all his shots were 3-pointers?

**Problem 10.** Let \( S = \{1, 2, 3, \ldots, 2019\} \). Find the number of non-constant functions \( f : S \rightarrow S \) such that

\[
f(k) = f(f(k + 1)) \leq f(k + 1)
\]

for all \( 1 \leq k \leq 2018 \).

Express your answer in the form \( \binom{m}{n} \), where \( m \) and \( n \) are integers.

**ANSWER TO PROBLEM 9**

**ANSWER TO PROBLEM 10**