2019 DUKE MATH MEET INDIVIDUAL ROUND

Problems 1-2 Name ____________________

Time Limit: 10 minutes Team ____________________

Problem 1. Compute the value of N, where

$$N = 818^3 - 6 \cdot 818^2 \cdot 209 + 12 \cdot 818 \cdot 209^2 - 8 \cdot 209^3.$$

Problem 2. Suppose $x \leq 2019$ is a positive integer that is divisible by 2 and 5, but not 3. If 7 is one of the digits in x, how many possible values of x are there?

ANSWER TO PROBLEM 1

ANSWER TO PROBLEM 2

2019 DUKE MATH MEET INDIVIDUAL ROUND

Problems 3-4 **Name** ____________________

Time Limit: 10 minutes **Team** ____________________

Problem 3. Find all non-negative integer solutions (a, b) to the equation

$$b^2 + b + 1 = a^2.$$

Problem 4. Compute the remainder when $\sum_{n=1}^{2019} n^4$ is divided by 53.

ANSWER TO PROBLEM 3

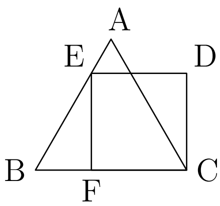
ANSWER TO PROBLEM 4

2019 DUKE MATH MEET INDIVIDUAL ROUND

Problems 5-6 **Name** ________________

Time Limit: 10 minutes **Team** ________________

Problem 5. Let ABC be an equilateral triangle and $CDEF$ a square such that E lies on segment AB and F on segment BC. If the perimeter of the square is equal to 4, what is the area of triangle ABC?



Problem 6.

$$S = \frac{4}{1 \times 2 \times 3} + \frac{5}{2 \times 3 \times 4} + \frac{6}{3 \times 4 \times 5} + \cdots + \frac{101}{98 \times 99 \times 100},$$

Let $T = \frac{5}{4} - S$. If $T = \frac{m}{n}$, where m and n are relatively prime integers, find the value of $m + n$.

ANSWER TO PROBLEM 5

ANSWER TO PROBLEM 6

2019 DUKE MATH MEET INDIVIDUAL ROUND

Problems 7-8 Name ___________________

Time Limit: 10 minutes Team ___________________

Problem 7. Find the sum of

$$\sum_{i=0}^{2019} \frac{2^i}{2^i + 2^{2019-i}}.$$

Problem 8. Let A and B be two points in the Cartesian plane such that A lies on the line $y = 12$, and B lies on the line $y = 3$. Let C_1, C_2 be two distinct circles that intersect both A and B and are tangent to the x-axis at P and Q, respectively. If $PQ = 420$, determine the length of AB.

ANSWER TO PROBLEM 7

ANSWER TO PROBLEM 8

2019 DUKE MATH MEET INDIVIDUAL ROUND

Problems 9-10 Name ___________________

Time Limit: 10 minutes Team ___________________

Problem 9. Zion has an average 2 out of 3 hit rate for 2-pointers and 1 out of 3 hit rate for 3-pointers. In a recent basketball match, Zion scored 18 points without missing a shot, and all the points came from 2 or 3-pointers. What is the probability that all his shots were 3-pointers?

Problem 10. Let $S = \{1, 2, 3, \ldots, 2019\}$. Find the number of non-constant functions $f : S \to S$ such that

$$f(k) = f(f(k+1)) \leq f(k+1) \quad \text{for all } 1 \leq k \leq 2018.$$

Express your answer in the form $\binom{m}{n}$, where m and n are integers.

ANSWER TO PROBLEM 9

ANSWER TO PROBLEM 10