Power Round: Semi-geometric Sequences

Duke Math Meet

November 2018

During this round, we will explore some special types of sequences. Some of the later problems may need earlier results, so feel free to use any previous parts even if you are not able to solve them yet. There are fifty possible points you can earn from this round.

So, without further ado, please enjoy!

1 SG Sequences

Definition 1. Let $a_1, a_2, \cdots, a_n$ be a sequence of positive integers. We say that the sequence is **semi-geometric** if for all $i = 1, 2, \cdots, n-1$, there exists an integer $k_i > 1$ such that $k_i a_i = a_{i+1}$. For instance, $2, 6, 12, 24$ is semi-geometric, whereas $2, 3, 6, 12$ is not since 2 does not divide 3. We call all such sequences semi-geometric sequences, or **SG sequences**.

Now that we've defined what SG sequences are, we will do some exercises that will help you better understand the concept.

Problem 1 (4 points, 2 points each)**.** Answer the following questions.

(a) Compute all SG sequences of length 5 such that $a_5 = 84$.

(b) Compute all SG sequences of length 3 such that $a_3 = 36$.

Problem 2 (4 points, 2 points each)**.** Let $a_1, \cdots, a_n$ be an SG sequence such that $a_n = 50400$.

(a) What is the largest possible value for n? (In other words, what is the length of the longest possible sequence with last term equal to 50400?)

(b) Let k be the answer you receive from part a. What is the number of distinct SG sequences of length k such that $a_k = 50400$?

Problem 3 (3 points)**.** Find the number of distinct SG sequences whose last term equals 420. (Note: the sequence (420) is also an SG sequence of length 1)

2 SG Numbers

In this section, we will explore some interesting properties of such SG sequences, namely the *SG numbers*.

Definition 2. For each positive integer n, we define the **SG number** of n to be the length of the longest SG sequence $a_1, \cdots, a_k$ such that $a_1 + \cdots + a_k = n$. In this particular case, we denote $\sigma(n) = k$.

(For instance, $\sigma(9) = 3$ since the SG sequence $1, 2, 6$ adds upto 9, whereas there cannot be an SG sequence with length 4 that add up to 9. This is because for an SG sequence to add upto 9, its first term must divide 9, hence it must be either 1, 3, or 9. If it's 9, it has length of 1. If it's 3, the only possible SG sequence is $3, 6$, which has length of 2. If it's 1, then the remaining sequence can be either 8 or $2, 6$, which implies that $1, 2, 6$ is the longest SG sequence that adds up to 9, hence $\sigma(9) = 3$.)

Here are some exercises that will help us familiarize ourselves with SG numbers.

Problem 4 (12 points, 3 points each). Find, with proof, the following values:

(a) $\sigma(19)$.

(b) $\sigma(25)$.

(c) $\sigma(95)$

(d) $\sigma(100)$.

Problem 5 (4 points). Find, with proof, the value of $\sigma(360)$ and all SG sequences with length $\sigma(360)$ that sum to 360.

3 Properties of SG Numbers

In this section, we will prove some general properties of SG numbers.

Problem 6 (2 points). Prove that there exists a positive integer n such that there are five distinct SG sequences with length $\sigma(n)$ that sum to n.

Problem 7 (3 points). For any positive integer k, what is the smallest integer n such that $\sigma(n) = k$?

Problem 8 (3 points). If a positive integer n has k digits in its binary (base-2) representation with $k-1$ ones and 1 zero, prove that $\sigma(n) = k-1$.

Problem 9 (4 points). Find, with proof, the second **and** the third smallest integers a, b such that $\sigma(a) = \sigma(b) = 10$.

Problem 10 (4 points, 2 points each). Answer the following questions.

(a) Prove that for all integers $a, b > 1$, $\sigma(ab+1) > \sigma(a)$.

(b) Prove that for all integers $a, b > 1$, $\sigma(ab) \geq \sigma(a)$. Provide an example where the equality holds.

Problems above induce some interesting thoughts about SG numbers. In the next few problems, we will try to come up with some upper bounds with regards to some particular SG numbers, namely 2 and 3.

Problem 11 (5 points). Find the largest integer n such that $\sigma(n) = 2$, and prove that any integer greater than n must have SG number of greater than 2.

The following two problems are only worth one point each. This is not because the following problems are necessarily any easier than the others. In fact, it's rather the opposite. We strongly advise that the students work on previous problems first and make sure they have everything correct before they dive into the last two.

Problem 12 (1 point). Prove that for all $n \geq 3$, the number $24n$ has SG number of greater than 3.

Problem 13 (1 point). Prove that for all $n > 48$ such that 24 does not divide n, $\sigma(n) > 3$.

Problemwriter's remark: With the last two problems, we may conclude that $\sigma(n) > 3$, for all $n > 48$. And in fact, some computer program results have made the problemwriter conjecture that for any integer k, there are only finitely many integers with SG number equal to k. Although this is an exciting result that we would love to have the students to solve, the DMM has decided against it, as it would not be very nice to ask the students to complete such a task within one hour.

We hope you enjoyed the topic!