Individual Round

Duke Math Meet 2018

November 2018

Problem 1. Let $f(x) = \frac{3x^3+7x^2-12x+2}{x^2+2x-3}$. Find all integers n such that $f(n)$ is an integer.

Problem 2. How many ways are there to arrange 10 trees in a line where every tree is either a yew or an oak and no two oak trees are adjacent?

Problem 3. 20 students sit in a circle in a math class. The teacher randomly selects three students to give a presentation. What is the probability that none of these three students sit next to each other?

Problem 4. Let $f_0(x) = x + |x-10| - |x+10|$, and for $n \geq 1$, let $f_n(x) = |f_{n-1}(x)| - 1$. For how many values of x is $f_{10}(x) = 0$?

Problem 5. 2 red balls, 2 blue balls, and 6 yellow balls are in a jar. Zion picks 4 balls from the jar at random. What is the probability that Zion picks at least 1 red ball and 1 blue ball?

Problem 6. Let $\triangle ABC$ be a right-angled triangle with $\angle ABC = 90^\circ$ and $AB = 4$. Let D on $\overline{AB}$ such that $AD = 3DB$ and $\sin \angle ACD = \frac{3}{5}$. What is the length of BC?

Problem 7. Find the value of

$$\frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{1+\cdots}}}}}$$

Problem 8. Consider all possible quadrilaterals $\square ABCD$ that have the following properties; $\square ABCD$ has integer side lengths with $AB||CD$, the distance between $\overline{AB}$ and $\overline{CD}$ is 20, and $AB = 18$. What is the maximum area among all these quadrilaterals, minus the minimum area?

Problem 9. How many perfect cubes exist in the set $\{1^{2018}, 2^{2017}, 3^{2016}, \ldots, 2017^2, 2018^1\}$?

Problem 10. Let n be the number of ways you can fill a 2018×2018 array with the digits 1 through 9 such that for every 11×3 rectangle (not necessarily for every 3×11 rectangle), the sum of the 33 integers in the rectangle is divisible by 9. Compute $\log_3 n$.