Duke Math Meet 2017
Relay Round Solution

Relay Round 1

1. We have

\[
\sin 20^{\circ} \cdot \sin 10^{\circ} \cdot \sin 50^{\circ} \cdot \sin 70^{\circ} = \sin 20^{\circ} \cos 20^{\circ} \cdot \cos 40^{\circ} \cdot \cos 80^{\circ}
\]
\[
= \frac{1}{2} \sin 40^{\circ} \cdot \cos 40^{\circ} \cdot \cos 80^{\circ}
\]
\[
= \frac{1}{4} \sin 80^{\circ} \cdot \cos 80^{\circ}
\]
\[
= \frac{1}{8} \sin 160^{\circ} = \frac{1}{8} \sin 20^{\circ}
\]

Therefore \( \sin 10^{\circ} \sin 30^{\circ} \sin 50^{\circ} \sin 70^{\circ} = \frac{1}{16} \). The answer is \( 16 \).

2. \( T = 16 \). We have

\[
a_{n+1} - 4a_n = 4(a_n - 4a_{n-1}) = 4^2(a_{n-1} - 4a_{n-2}) = \cdots = 4^n(a_1 - 4a_0) = 0
\]

So we have \( a_{n+1} = 4a_n \). Therefore \( a_n = 4^n \) and

\[
\log_2 a_{2017} = 4034
\]

3. \( T = 4034 \). We want to show, by induction, that if there are \( n = 2k \) participants, the largest total number of matches played in the tournament is less than \( k^2 \). First, when \( k = 1 \), there are only two participants, so the total number of matches can’t exceed 1. Therefore the statement is true. Suppose it is true for \( k - 1 \), then given a tournament with \( 2k \) participants, we can find a pair that has played a match. Let’s say participant \( A \) played with participant \( B \). If they together played more than \( n - 1 \) matches in total, then by pigeonhole principle, there exists another participant \( C \) who played with both \( A \) and \( B \). It contradicts to the fact that no 3 participants of whom each pair has played with each other. So the total number of matches played by \( A \) and \( B \) is at most \( n - 1 \). The rest \( 2n - 2 \) participants played at most \( (k - 1)^2 \) matches by induction hypothesis. Therefore the total number of matches in this tournament is at most

\[
(k - 1)^2 + n - 1 = (k - 1)^2 + 2k - 1 = k^2.
\]

Taking \( n = 4034 = 2k \), we obtain the answer \( 2017^2 = 4068289 \).
1. We have \( p = (c^2 - q)(c^2 + q) \). Since \( p \) is a prime, \( c^2 - q = 1 \) and \( c^2 + q = p \). Now we know \( q = c^2 - 1 = (c - 1)(c + 1) \). So either \( c - 1 = 1 \) and \( c + 1 = q \) or \( c - 1 = -q \) and \( c + 1 = -1 \). Both solutions show that \( q = 3 \). Then \( p = 7 \). Hence \( p + q = 10 \).

2. \( T = 10 \). Let \( s_n \) be the number of subsets in \( \{1, 2, \ldots, n\} \) that do not contain two consecutive numbers. Then \( s_n = s_{n-1} + s_{n-2} \). We know \( s_1 = 2 \) and \( s_2 = 3 \). So \( s_{10} = 144 \).

3. \( T = 144 \). Since \( a + b + c = 0 \), \( a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) = 0 \). Hence, \( abc = \frac{-6}{3} \).