Power Round Solution

Duke Math Meet 2017

Problem 1.

We name these six people A, B, C, D, E, F. By pigeonhole principle, A has either three friends or three strangers among the other five people. Suppose A has three friends. Then without the loss of generality, we can assume B, C, D are A's friends. If any two of these three people are friends with each other, then together with A, they form a group of three of which every two are friends. If no two of these three people are friends, then B, C, D form a group of three of which no two are friends. Our statement follows either way. Similar argument if A has three strangers.

Problem 2.

Let c be a coloring function on K_n Define $f : c \mapsto G$ such that $f(c) = G = (V, E)$ where $E = \{e \in E_K \mid c(e) = \text{red}\}$. It is surjective since any subgraph $G' = (V, E' \subseteq E_K)$ of K_n we can have a coloring which color the edges in E' with red and the rest edges in E_K with blue. It is a valid coloring, so f maps this coloring function to G'. The map f is also injective. To see that suppose $f(c_1) = f(c_2)$, then two subgraphs $f(c_1)$ and $f(c_2)$ has the same edge set. So two coloring c_1 and c_2 coincide on these edges. But on all the other edges, the color is blue. So two coloring are the same on all edges. So the map is one-to-one.

Problem 3.

Let c be a coloring on K_n. Then we denote $\bar{c}$ be the opposite coloring function of c. That is, for any edge e, $\bar{c}(e) = \text{blue}$ when $c(e) = \text{red}$ and $\bar{c}(e) = \text{red}$ when $c(e) = \text{blue}$. We can easily check that $c \mapsto \bar{c}$ is a bijective map on all coloring functions of K_n.

Suppose any coloring c on K_n yields either a red K_s or blue K_t, then $\bar{c}$ yields either a red K_t or a blue K_s. This is true for all coloring $\bar{c}$. Thus, $R(t, s) \leq R(s, t)$.

To establish equality, let $m = R(s, t) - 1$. Then by definition, some coloring b on K_m does not contain any red K_s or blue K_t. Then $\bar{b}$ does not contain any red K_t or blue K_s. This implies that $R(t, s) > m = R(s, t) - 1$. Therefore they must equal.

Problem 4.

(i) See figure 1. Dashed lines are red edges and solid lines are blue edges.

(ii) By Theorem 1.9, $R(3, 3) \leq 6$. By part (i), $R(3, 3) > 5$. Therefore $R(3, 3) = 6$.

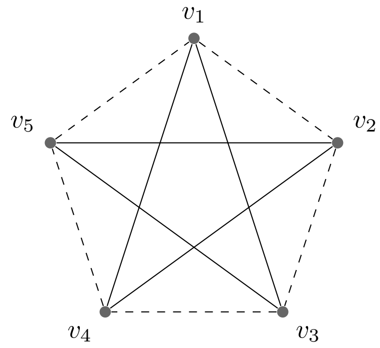

Figure 1: Problem 4 (i)

Problem 5.

(i) Let $n = R(s,t) \leq N$. For any complete graph K_N, we pick any n vertices and they form a complete subgraph K_n. For any coloring c on K_N, we have a unique coloring $\bar{c}$ on K_n that coincides with c on the edges of K_n. Then, by definition of Ramsey number, this coloring will yields either a red K_s or a blue K_t. This is true for any coloring on K_N, so our statement follows.

(ii) Let $N = R(s-1,t) + R(s,t-1)$. Fix a vertex A in K_N. Then A has $N-1$ edges incident to it. By pigeonhole principle, A has either $R(s-1,t)$ red edges or $R(s,t-1)$ blue edges. If it has $R(s-1,t)$ red edges, then the endpoints of these edges forms a subgraph $K_{R(s-1,t)}$. So for any 2-coloring, it contains a red K_{s-1} or a blue K_t. If it is the second case, we are done. If it is the first case, the red K_{s-1} and A forms a red K_s. So either way there's either a red K_s or a blue K_t. Similar arguments goes for the case where A has $R(s,t-1)$ blue edges.

(iii) Induction on $n = s+t$. It is trivial for base case where $n = 2$ since $R(1,s) = 1$ for any $s \in \mathbb{N}$. Suppose the statement is true for k. Then for $s+t = k+1$, we have $R(s-1,t)$ and $R(s,t-1)$ are finite by induction hypothesis. Then apply part (ii), we have $R(s,t) \leq R(s-1,t) + R(s,t-1)$. Hence $R(s,t)$ is finite.

(iv) Again we use induction base on $n = s+t$. The base cases are easy to verify by problem 3 (i). Suppose the inequality is true for all s and t such that $k = s+t$. Then for all s,t where $s+t = k+1$, we have

$$\begin{aligned} R(s,t) &\leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} \\ &= \frac{(s+t-3)!}{(s-2)!(t-1)!} + \frac{(s+t-3)!}{(s-1)!(t-2)!} \\ &= \frac{(s+t-3)!(s-1+t-1)}{(s-1)!(t-1)!} = \binom{s+t-2}{s-1} \end{aligned}$$

(v) For any $k \in \mathbb{N}$, $4k^2 \geq 2k \cdot (2k-1)$. So $4^n(n!)^2 \geq (2n)!$. Then,

$$\binom{2s-2}{s-1} = \frac{(2s-2)!}{(s-1)!(s-1)!} \leq 2^{2s-2}$$

Problem 6.

(i) Let v_1, ..., v_{3s-1} be $3s-1$ vertices. Color the edge v_i and v_j with red if $i-j \in \{s, s+1, ..., 2s-1\} \pmod{3s-1}$ and color all other edges with blue. Notice that $3s-1 < 3s$ and $3 \cdot (2s-1) < 2 \cdot (3s-1)$. Then for any distinct i, j, k, we have $(i-j)+(j-k)+(k-i) = 0 \pmod{3s-1}$. Therefore at least one of $i-j$, $j-k$ and $k-i$ is either less than s or greater than $2s-1$. So one of the edges among vertices v_i, v_j and v_k are colored blue. It then follows that this coloring contains no red K_3.

Now it suffices to show that no blue K_{s+1} exists for this coloring. Suppose there is a blue $K = K_{s+1}$, then we can assume that v_1 is in the subgraph K. Accordingly, v_{s+1}, ..., v_{2s} can't be in the subgraph K. So in the set $P = \{v_2, ..., v_s, v_{2s+1}, ..., v_{3s-1}\}$, we need to have s vertices in K. Let edge $e_i = \{v_{i+1}, v_{2s+i}$ for $i = 1, ..., s-1$. Then the e_i is colored red. There are $s-1$ red edges but we need to pick s vertices in the set P. By pigeonhole principle, e_i is in K for some i. Then K can't be a blue K_{s+1}.

By this construction, there's a coloring on K_{3s-1} such that it contains neither red K_3 nor blue K_{s+1}. Hence $R(3, s+1) > 3s-1$.

(ii) Part (i) showed that $R(3, 4) > 8$. So we only need to verify that any 2-coloring on K_9 yields a red K_3 or a blue K_4. Fix a vertex A. We know that A has 8 edges. Then we have three situations.

Suppose at least 6 of them are blue, then the endpoints of blue edges form a K_6. By Theorem 1.9, it has either a red K_3 or a blue K_3. Either case, together with A, the graph contains either a red K_3 or a blue K_4.

Suppose at most 4 of them are blue. In other word, at least 4 edges incident to A are red. Then if any pair of the endpoints has a red edge, we are done. So all end points are connected with each other by blue edges. But that gives us a blue K_4.

The only case that hasn't been verified is that 3 edges of A are red and 5 are blue. Now we consider other vertices. It turns out that if any vertex doesn't have exactly 3 red edges, by previous discussion, we can find either a red K_3 or a blue K_4. So there are 27 red edges in total if add up the number of red edges on all 9 nodes. But notice that every edge has two endpoints, which means they were counted twice. So we have 13.5 red edges in K_9. That leads to a contradiction. So at least one vertex doesn't have exactly 3 red edges. This concludes our proof.

Problem 7.

Suppose we have four points $a, b, c, d \in \mathbb{Z}_{17}$. Without the loss of generality, we can assume that $0 \le a < b < c < d \le 16$. In fact, we can also assume $a = 0$, otherwise we will just rotate the numbering to make $a = 0$.

Suppose they a red K_4, that means $i - j = \pm 1,\ \pm 2,\ \pm 4,\ \pm 8 \pmod{17}$ for all $i, j \in \{a, b, c, d\}$. Let $S = \{1, 2, 4, 8, 9, 13, 15, 16\}$. Then $b, c, d, d-c, c-b, d-b \in S$. A straightforward search will show that no such b, c, d exist. Do the similar to $\bar{S}$ will prove that no monochromatic complete subgraph in K_{17}.

There's a relatively easier approach: observe that $\mathbb{Z}_{17}$ is a field, which implies that every element has an inverse. But S is closed under multiplication, so we can multiply b^{-1} to $b, c, d, d-c, d-b, c-b$. Therefore we just need to set $b = 1$ and verify for all c, d in S and in $\bar{S}$.

Either way, this graph shows that $R(4,4) > 17$. By previous problems, $R(4,4) \leq 2 * R(3,4) = 18$. Hence $R(4,4) = 18$.

Problem 8.

(i) Let $n = R_{r-1}(R(s_1, s_2), s_3, ..., s_r)$. For any r-coloring c on K_n, if we view color 1 and color 2 as the same color, then c gives a $(r-1)$-coloring. As a result, there exists either a K_{s_i} of color $i \geq 3$ or a $K_{R(s_1,s_2)}$ of color 1 and 2. If it is the first case, then we are done. If it is the second case, by definition, this subgraph contains either a K_{s_1} of color 1 or a K_{s_2} of color 2. Therefore any r-coloring on K_n contains some K_{s_i} of color i. By the minimality Ramsey numbers, we can conclude that $R_r(s_1, ..., s_r) \leq n = R_{r-1}(R(s_1, s_2), s_3, ..., s_r)$

(ii) $R_3(3,3,3) \leq R(R(3,3),3) = R(6,3) \leq \binom{7}{2} = 21$