Problem 1.

We name these six people A, B, C, D, E, F. By pigeonhole principle, A has either three friends or three strangers among the other five people. Suppose A has three friends. Then without the loss of generality, we can assume B, C, D are A’s friends. If any two of these three people are friends with each other, then together with A, they form a group of three of which every two are friends. If no two of these three people are friends, then B, C, D form a group of three of which no two are friends. Our statement follows either way. Similar argument if A has three strangers.

Problem 2.

Let c be a coloring function on $K_n$. Define $f : c \mapsto G$ such that $f(c) = G = (V, E)$ where $E = \{e \in E_K \mid c(e) = \text{red}\}$. It is surjective since any subgraph $G' = (V, E' \subseteq E_K)$ of $K_n$ we can have a coloring which color the edges in $E'$ with red and the rest edges in $E_K$ with blue. It is a valid coloring, so $f$ maps this coloring function to $G'$. The map $f$ is also injective. To see that suppose $f(c_1) = f(c_2)$, then two subgraphs $f(c_1)$ and $f(c_2)$ has the same edge set. So two coloring $c_1$ and $c_2$ coincide on these edges. But on all the other edges, the color is blue. So two coloring are the same on all edges. So the map is one-to-one.

Problem 3.

Let c be a coloring on $K_n$. Then we denote $\bar{c}$ be the opposite coloring function of c. That is, for any edge $e$, $\bar{c}(e) = \text{blue}$ when $c(e) = \text{red}$ and $\bar{c}(e) = \text{red}$ when $c(e) = \text{blue}$. We can easily check that $c \mapsto \bar{c}$ is a bijective map on all coloring functions of $K_n$.

Suppose any coloring c on $K_n$ yields either a red $K_s$ or blue $K_t$, then $\bar{c}$ yields either a red $K_t$ or a blue $K_s$. This is true for all coloring $\bar{c}$. Thus, $R(t, s) \leq R(s, t)$.

To establish equality, let $m = R(s, t) - 1$. Then by definition, some coloring $b$ on $K_m$ does not contain any red $K_s$ or blue $K_t$. Then $\bar{b}$ does not contain any red $K_t$ or blue $K_s$. This implies that $R(t, s) > m = R(s, t) - 1$. Therefore they must equal.

Problem 4.

(i) See figure 1. Dashed lines are red edges and solid lines are blue edges.

(ii) By Theorem 1.9, $R(3, 3) \leq 6$. By part (i), $R(3, 3) > 5$. Therefore $R(3, 3) = 6$. 
Problem 5.

(i) Let \( n = R(s, t) \leq N \). For any complete graph \( K_N \), we pick any \( n \) vertices and they form a complete subgraph \( K_n \). For any coloring \( c \) on \( K_N \), we have a unique coloring \( \bar{c} \) on \( K_n \) that coincides with \( c \) on the edges of \( K_n \). Then, by definition of Ramsey number, this coloring will yield either a red \( K_s \) or a blue \( K_t \). This is true for any coloring on \( K_N \), so our statement follows.

(ii) Let \( N = R(s - 1, t) + R(s, t - 1) \). Fix a vertex \( A \) in \( K_N \). Then \( A \) has \( N - 1 \) edges incident to it. By pigeonhole principle, \( A \) has either \( R(s - 1, t) \) red edges or \( R(s, t - 1) \) blue edges. If it has \( R(s - 1, t) \) red edges, then the endpoints of these edges form a subgraph \( K_{R(s - 1, t)} \). So for any 2-coloring, it contains a red \( K_{s - 1} \) or a blue \( K_t \). If it is the second case, we are done. If it is the first case, the red \( K_{s - 1} \) and \( A \) forms a red \( K_s \). So either way there’s either a red \( K_s \) or a blue \( K_t \). Similar arguments goes for the case where \( A \) has \( R(s, t - 1) \) blue edges.

(iii) Induction on \( n = s + t \). It is trivial for base case where \( n = 2 \) since \( R(1, s) = 1 \) for any \( s \in \mathbb{N} \). Suppose the statement is true for \( k \). Then for \( s + t = k + 1 \), we have \( R(s - 1, t) \) and \( R(s, t - 1) \) are finite by induction hypothesis. Then apply part (ii), we have \( R(s, t) \leq R(s - 1, t) + R(s, t - 1) \). Hence \( R(s, t) \) is finite.

(iv) Again we use induction base on \( n = s + t \). The base cases are easy to verify by problem 3 (i). Suppose the inequality is true for all \( s \) and \( t \) such that \( k = s + t \). Then for all \( s, t \) where \( s + t = k + 1 \), we have

\[
R(s, t) \leq \binom{s + t - 3}{s - 2} + \binom{s + t - 3}{s - 1}
\]

\[
= \frac{(s + t - 3)!}{(s - 2)!(t - 1)!} + \frac{(s + t - 3)!}{(s - 1)!(t - 2)!}
\]

\[
= \frac{(s + t - 3)!(s - 1 + t - 1)}{(s - 1)!(t - 1)!} = \binom{s + t - 2}{s - 1}
\]

(v) For any \( k \in \mathbb{N} \), \( 4k^2 \geq 2k \cdot (2k - 1) \). So \( 4^n(n!)^2 \geq (2n)! \). Then,

\[
\binom{2s - 2}{s - 1} = \frac{(2s - 2)!}{(s - 1)!(s - 1)!} \leq 2^{2s - 2}
\]
Problem 6.

(i) Let \( v_1, \ldots, v_{3s-1} \) be \( 3s - 1 \) vertices. Color the edge \( v_i \) and \( v_j \) red if \( i - j \in \{s, s + 1, \ldots, 2s - 1\} \) (mod \( 3s - 1 \)) and color all other edges with blue. Notice that \( 3s - 1 < 3s \) and \( 3 \cdot (2s - 1) < 2 \cdot (3s - 1) \). Then for any distinct \( i, j, k \), we have \( (i - j) + (j - k) + (k - i) = 0 \) (mod \( 3s - 1 \)). Therefore at least one of \( i - j, j - k \) and \( k - i \) is either less than \( s \) or greater than \( 2s - 1 \). So one of the edges among vertices \( v_i, v_j \) and \( v_k \) are colored blue. It then follows that this coloring contains no red \( K_3 \).

Now it suffices to show that no blue \( K_{s+1} \) exists for this coloring. Suppose there is a blue \( K = K_{s+1} \), then we can assume that \( v_1 \) is in the subgraph \( K \). Accordingly, \( v_{s+1}, \ldots, v_{2s} \) can’t be in the subgraph \( K \). So in the set \( P = \{v_2, \ldots, v_s, v_{2s+1}, \ldots, v_{3s-1}\} \), we need to have \( s \) vertices in \( K \). Let edge \( e_i = \{v_{i+1}, v_{2s+i}\} \) for \( i = 1, \ldots, s - 1 \). Then the \( e_i \) is colored red. There are \( s - 1 \) red edges but we need to pick \( s \) vertices in the set \( P \). By pigeonhole principle, \( e_i \) is in \( K \) for some \( i \). Then \( K \) can’t be a blue \( K_{s+1} \).

By this construction, there’s a coloring on \( K_{3s-1} \) such that it contains neither red \( K_3 \) nor blue \( K_{s+1} \). Hence \( R(3, s + 1) > 3s - 1 \).

(ii) Part (i) showed that \( R(3, 4) > 8 \). So we only need to verify that any 2-coloring on \( K_9 \) yields a red \( K_3 \) or a blue \( K_4 \). Fix a vertex \( A \). We know that \( A \) has 8 edges. Then we have three situations.

Suppose at least 6 of them are blue, then the endpoints of blue edges form a \( K_6 \). By Theorem 1.9, it has either a red \( K_3 \) or a blue \( K_3 \). Either case, together with \( A \), the graph contains either a red \( K_3 \) or a blue \( K_4 \).

Suppose at most 4 of them are blue. In other word, at least 4 edges incident to \( A \) are red. Then if any pair of the endpoints has a red edge, we are done. So all end points are connected with each other by blue edges. But that gives us a blue \( K_4 \).

The only case that hasn’t been verified is that 3 edges of \( A \) are red and 5 are blue. Now we consider other vertices. It turns out that if any vertex doesn’t have exactly 3 red edges, by previous discussion, we can find either a red \( K_3 \) or a blue \( K_4 \). So there are 27 red edges in total if add up the number of red edges on all 9 nodes. But notice that every edge has two endpoints, which means they were counted twice. So we have 13.5 red edges in \( K_9 \). That leads to a contradiction. So at least one vertex doesn’t have exactly 3 red edges. This concludes our proof.

Problem 7.

Suppose we have four points \( a, b, c, d \in \mathbb{Z}_{17} \). Without the loss of generality, we can assume that \( 0 \leq a < b < c < d \leq 16 \). In fact, we can also assume \( a = 0 \), otherwise we will just rotate the numbering to make \( a = 0 \).

Suppose they a red \( K_4 \), that means \( i - j = \pm 1, \pm 2, \pm 4, \pm 8 \) (mod 17) for all \( i, j \in \{a, b, c, d\} \). Let \( S = \{1, 2, 4, 8, 9, 13, 15, 16\} \). Then \( b, c, d, d - c, c - b, d - b \in S \). A straightforward search will show that no such \( b, c, d \) exist. Do the similar to \( S \) will prove that no monochromatic complete subgraph in \( K_{17} \).

There’s a relatively easier approach: observe that \( \mathbb{Z}_{17} \) is a field, which implies that every element has an inverse. But \( S \) is closed under multiplication, so we can multiply \( b^{-1} \) to \( b, c, d, d - c, c - b, c - b \). Therefore we just need to set \( b = 1 \) and verify for all \( c, d \in S \) and in \( S \).
Either way, this graph shows that \( R(4, 4) > 17 \). By previous problems, \( R(4, 4) \leq 2 \cdot R(3, 4) = 18 \). Hence \( R(4, 4) = 18 \).

**Problem 8.**

(i) Let \( n = R_{r-1}(R(s_1, s_2), s_3, ..., s_r) \). For any \( r \)-coloring \( c \) on \( K_n \), if we view color 1 and color 2 as the same color, then \( c \) gives a \((r - 1)\)-coloring. As a result, there exists either a \( K_{s_i} \) of color \( i \geq 3 \) or a \( K_{R(s_1, s_2)} \) of color 1 and 2. If it is the first case, then we are done. If it is the second case, by definition, this subgraph contains either a \( K_{s_1} \) of color 1 or a \( K_{s_2} \) of color 2. Therefore any \( r \)-coloring on \( K_n \) contains some \( K_{s_i} \) of color \( i \). By the minimality Ramsey numbers, we can conclude that \( R_r(s_1, ..., s_r) \leq n = R_{r-1}(R(s_1, s_2), s_3, ..., s_r) \)

(ii) \( R_3(3, 3, 3) \leq R(R(3, 3), 3) = R(6, 3) \leq \binom{7}{2} = 21 \)