Duke Math Meet 2016

Relay Solutions

1.1 From the first sequence, we have $y - x = 4(a_2 - a_1)$. From the second relationship, we have $3(b_4 - b_3) = 2(y - x)$. Hence we have $\frac{b_4 - b_3}{a_2 - a_1} = \frac{8}{3}$. Hence $p + q = 8 + 3 = \boxed{11}$.

1.2 $T = 11$. We have $(x^2 - 7x + 11)^{x^2-1} = 1$. $a^b = 1$ when $a = 1$, $b = 0$, or $a = -1$ and b is even. $x^2 - 7x + 11 = 1 \implies x = 2, 5$. $x^2 - 1 = 0$ when $x = 1, -1$. $x^2 - 7x + 11 = -1 \implies x = 3, 4$ but $x^2 - 1$ is only even when $x = 3$. So we have a total of $\boxed{5}$ solutions which are $-1, 1, 2, 3, 5$.

1.3 $TNYWR = 5$. Let a_n be the probability that David has the ball after n seconds. If David has the ball after n seconds, then he can't have the ball after $n + 1$ seconds. If he doesn't have the ball, then there's a $\frac{1}{5}$ the ball can be passed to him. Hence $a_{n+1} = \frac{1}{5}(1 - a_n)$. Starting with initial condition of $a_0 = 1$, we see that $a_1 = 0, a_2 = \frac{1}{5}, a_3 = \frac{4}{25}, a_4 = \frac{21}{125}, a_5 = \boxed{\frac{104}{625}}$.

2.1 $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - 2\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{12} = \boxed{\frac{\pi^2}{12}}$.

2.2 Hence $T = 12$. If we make a Venn diagram, we see that every toy must lie in one of the circles but cannot lie in the all three. So there are a total of 6 parts of the circle that each toy can go. Hence the answer is $\boxed{6^{12}}$.

2.3 We see that $T = 12$ and $n = 29$. a_{28} shakes with everyone; a_{27} shakes with everyone except a_1; a_{26} everyone except a_1, a_2; ... a_{15} everyone except $a_1, .., a_{13}$ then we are done. Hence, the answer is $\boxed{14}$.