Duke Math Meet 2015

Team Round

1. Let $U = \{-2, 0, 1\}$ and $N = \{1, 2, 3, 4, 5\}$. Let f be a function that maps U to N. For any $x \in U$, $x + f(x) + xf(x)$ is an odd number. How many f satisfy the above statement?

2. Around a circle are written all of the positive integers from 1 to n,$n \geq 2$ in such a way that any two adjacent integers have at least one digit in common in their decimal expressions. Find the smallest n for which this is possible.

3. Michael loses things, especially his room key. If in a day of the week he has n classes he loses his key with probability $n/5$. After he loses his key during the day he replaces it before he goes to sleep so the next day he will have a key. During the weekend(Saturday and Sunday) Michael studies all day and does not leave his room, therefore he does not lose his key. Given that on Monday he has 1 class, on Tuesday and Thursday he has 2 classes and that on Wednesday and Friday he has 3 classes, what is the probability that loses his key at least once during a week?

4. Given two concentric circles one with radius 8 and the other 5. What is the probability that the distance between two randomly chosen points on the circles, one from each circle, is greater than 7?

5. We say that a positive integer n is lucky if n^2 can be written as the sum of n consecutive positive integers. Find the number of lucky numbers strictly less than 2015.

6. Let $A = \{3^x + 3^y + 3^z | x, y, z \geq 0, x, y, z \in \mathbb{Z}, x < y < z\}$. Arrange the set A in increasing order. Then what is the 50th number? (Express the answer in the form $3^x + 3^y + 3^z$).

7. Justin and Oscar found 2015 sticks on the table. I know what you are thinking, that is very curious. They decided to play a game with them. The game is, each player in turn must remove from the table some sticks, provided that the player removes at least one stick and at most half of the sticks on the table. The player who leaves just one stick on the table loses the game. Justin goes first and he realizes he has a winning strategy. How many sticks does he have to take off to guarantee that he will win?

8. Let (x, y, z) with $x \geq y \geq z \geq 0$ be integers such that $\frac{x^3+y^3+z^3}{3} = xyz + 21$. Find x.

9. Let $p < q < r < s$ be prime numbers such that

$$1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} = \frac{1}{pqrs}$$

   Find $p + q + r + s$.

10. In "island-land", there are 10 islands. Alex falls out of a plane onto one of the islands, with equal probability of landing on any island. That night, the Chocolate King visits Alex in his sleep and tells him that there is a mountain of chocolate on one of the islands, with equal probability of being on each island. However, Alex has become very fat from eating chocolate his whole life, so he can't swim to any of the other islands. Luckily, there is a teleporter on each island. Each teleporter will teleport Alex to exactly one other teleporter (possibly itself) and each teleporter gets teleported to by exactly one teleporter. The configuration of the teleporters is chosen uniformly at random from all possible configurations of teleporters satisfying these criteria. What is the probability that Alex can get his chocolate?