1. a. $\frac{41}{24}$. Going reverse. $2+\frac{1}{3}=\frac{7}{3}, 2+\frac{3}{7}=\frac{17}{7}, 1+\frac{7}{17}=\frac{24}{17}, 1+\frac{17}{24}=\frac{41}{24}$

   b. $[1,2,3,4]$ $\frac{43}{30}=1+\frac{13}{30}, \frac{30}{13}=2+\frac{4}{13}, \frac{13}{4}=3+\frac{1}{4}, 4=4+0,$

   c. $[1,\overline{1,2}]$

$$\begin{aligned}\sqrt{3} &= 1+(\sqrt{3}-1)\\ \frac{1}{\sqrt{3}-1} &= \frac{\sqrt{3}+1}{2}\\ \frac{\sqrt{3}+1}{2} &= 1+\frac{\sqrt{3}-1}{2}\\ \sqrt{3}+1 &= \frac{2}{\sqrt{3}-1}\\ \sqrt{3}+1 &= 2+\sqrt{3}-1\end{aligned}$$

   Here we note that, we already encountered $\sqrt{3}-1$ (remainder of the first step) so we see that continued fraction will repeat.

   d. $\sqrt{6}$
   Let $x=[\overline{2,4}]$. Then we have $x=[2,4,x]$

$$\begin{aligned}x &= [2,4,x]\\ x &= [2,4+\frac{1}{x}]\\ x &= [2,\frac{4x+1}{x}]\\ x &= 2+\frac{x}{4x+1}\\ x &= \frac{9x+2}{4x+1}\end{aligned}$$

   So we see that $4x^2+x=9x+2 \implies 2x^2-4x-1=0$. So $x=\frac{4\pm2\sqrt{6}}{4}=\frac{2\pm\sqrt{6}}{2}$. Remember that we want $2+\frac{1}{x}=2+\sqrt{6}-2=\sqrt{6}$.

2. a. We prove by induction. Base case: $n=1$. $p_1q_0-p_0q_1=(a_1a_0+1)(1)-a_0a_1=1=(-1)^0$. Now suppose the claim is true for n.

$$\begin{aligned}p_{n+1}q_n-p_nq_{n+1} &= (a_{n+1}p_n+p_{n-1})q_n-p_n(a_{n+1}q_n+q_{n-1})\\ &= p_{n-1}q_n-p_nq_{n-1}\\ &= (-1)(-1)^{n-1}\\ &= (-1)^n\end{aligned}$$

   b. Note we used the identity from part a.

$$\begin{aligned}p_{n+1}q_{n-1}-p_{n-1}q_{n+1} &= (a_np_n+p_{n-1})q_{n-1}-p_{n-1}(aq_n+q_{n-1})\\ &= a_n(p_nq_{n-1}-p_{n-1}q_n)\\ &= a_n(-1)^{n-2}\\ &= (-1)a_n\end{aligned}$$

3. We first find the continued fraction of $\frac{1+\sqrt{5}}{2}$.

$$\begin{aligned}\frac{1+\sqrt{5}}{2} &= 1+\frac{\sqrt{5}-1}{2}\\ \frac{1+\sqrt{5}}{2} &= \frac{2}{\sqrt{5}-1}\end{aligned}$$

   So $\frac{1+\sqrt{5}}{2}=[\overline{1}]$. Hence all the $a_i=1$.

   Note that we have $p_n=p_{n-1}+p_{n-2}$ and $p_0=1=F_1, p_1=2=F_2$. Thus $p_n=F_{n+1}$. We also have $q_n=q_{n-1}+q_{n-2}$ and $q_0=1=F_0, q_1=1=F_1 \implies q_n=F_n$. So we have the nth convergent $\frac{p_{n-1}}{q_{n-1}}=\frac{F_n}{F_{n-1}}$

4. $(-34, 74)$ or anything of the form $(-34 + 567t, 74 - 1234t)$. We first find the continued fraction expansion $\frac{1234}{567} = [2, 5, 1, 2, 33]$.

a_n	2	5	1	2	33
p_n	2	11	13	37	1234
q_n	1	5	6	17	567

From problem we have that $1234(17) - 567(37) = -1$. So $1234(-34) + 567(74) = 2$.

5. a. From the given theorem, we know that α lies between any two consecutive convergents. So for $n \geq 1$, we have $|\alpha - \frac{p_n}{q_n}| < |\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}| < \frac{1}{q_{n+1}q_n}$.

   b. Suppose not. Then there exists n such that $|\alpha - \frac{p_n}{q_n}| < \frac{1}{2q_n^2}$ and $|\alpha - \frac{p_{n+1}}{q_{n+1}} < \frac{1}{2q_{n+1}^2}$. $|\alpha - \frac{p_n}{q_n}| + |\alpha - \frac{p_{n+1}}{q_{n+1}}| \geq |\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}| = \frac{1}{q_n q_{n+1}}$. But $\frac{1}{2q_{n+1}^2} + \frac{1}{2q_n^2} \geq \frac{1}{q_n q_{n+1}}$ by AM-GM inequality. Contradiction. Thus at least one of any two consecutive convergents to x works.

   c. We can see that $\sqrt{2} = [1, \overline{2}]$.

a_n	1	2	2	2	2	2
p_n	1	3	7	17	41	99
q_n	1	2	5	12	29	70

   $29 * 70 > 1000$ so we can use $\frac{41}{29}$ or $\frac{99}{70}$.

6. a. $\alpha - \alpha' > 0 \implies \frac{\sqrt{D}}{Q} > 0$ so $Q > 0$.
   $\alpha + \alpha' > 0$, so $P > 0$.
   $\alpha' < 0$ so $P < \sqrt{D}$
   $\alpha > 1$ so $Q < P + \sqrt{D} < 2\sqrt{D}$

   b. $\alpha - \lfloor\alpha\rfloor$ is of the form $\frac{A+\sqrt{D}}{B}$. Then $\frac{B}{A+\sqrt{D}} = \frac{-BA+\sqrt{D}B}{(D^2-A)}$. So α_1 is a quadratic irrational. Since $\alpha - \lfloor\alpha\rfloor < 1$, we know that $\alpha_1 > 1$. In addition, note that $\alpha' = \lfloor\alpha\rfloor + \frac{1}{\alpha_1'} \implies \alpha_1' = -\frac{1}{q_0 - \alpha'}$. α' is negative and q_0 is positive so $-1 < \alpha_1' < 0$. So α_1 is reduced.

   c. We know that inductively all the α_i are reduced. But there are a finite number of possibilities as seen by problem 6. Thus there exists $i < j$ such that $\alpha_i = \alpha_j$, and so $\alpha_{i+1} = \alpha_{j+1}$ and so on. So the sequence is periodic.

7. a. We want $\sqrt{N} + m > 1$ and $-1 < m - \sqrt{N} < 0$. So we see that $m = \lfloor\sqrt{N}\rfloor$.

   b. We know that $\sqrt{N} + m = [\overline{a_0, \ldots, a_n}]$. But note that $a_0 = \lfloor\sqrt{N} + m\rfloor = 2m$. So we have $\sqrt{N} + m = [\overline{2m, \ldots, a_n}]$ so $\sqrt{N} = [m, \overline{a_1, \ldots, a_n, 2m}]$.

8. a. The area would be $\frac{1}{2}|P_{n+2}Q_n - Q_{n+2}P_n| = \frac{a_{n+2}}{2}$ since triangle is half of the parallelogram. However since $P_{n+2} = a_{n+2}P_{n+1} + P_n$ and $Q_{n+2} = a_{n+2}Q_{n+1} + Q_n$. $(P_n + Q_n) + i(P_{n+1}, Q_{n+1})$ for $0 \leq i \leq a_{n+2}$ must all be boundary points. So we have $a_{n+2} + 2$ boundary points (don't forget to include origin). So by Pick's, we see that $I = 0$.

   (a) Area of parallelogram is $|2PQ - 2Q^2\alpha| = 2Q^2|\alpha - \frac{P}{Q}| < 1$. Now suppose there was an interior point in the region (x, y), then $(2P - x, 2Q - y)$ is also in the region. But then consider the figure $(0, 0), (x, y), (P, Q)$, and $(2P - x, 2Q - y)$. Pick's Theorem tells us that this area is at least 1. But this lies in the parallelogram. Contradiction.

   c. Suppose that $|a - \frac{P}{Q}| < \frac{1}{2Q^2}$ and $\frac{P}{Q}$ isn't a continued fraction convergent. Then from part b, we know no lattice points in that parallelogram. But then (P, Q) lies in the triangle of $(0, 0), (P_n, Q_n)$, and (P_{n+2}, Q_{n+2}). This can be seen as all three points lie on the same side of line thru $(2Q\alpha, 2Q)$ since by considering slope, we know (P_n, Q_n) and (P_{n+2}, Q_{n+2}) are on the same side and (P, Q) is between those two and thus must be on the same side. Contradiction to part a.