Duke Math Meet

Power Round

November 21, 2015

In the Power Round the entire team of six students will have 60 minutes to answer a series of proof-based questions. The team members may collaborate, but as with all rounds in the Duke Math Meet, only pen/pencil and paper may be used (no calculators of any kind). After 60 minutes the team will submit all solutions. Solutions to different numbered questions must be on different sheets of paper. Cross out anything you do not want graded. Teams will be given 30-minute, 5-minute, and 1-minute warnings. Teams may use results of previous problems to solve later problems, even if the team has not submitted solutions to those previous problems (this applies for parts of questions also) and should note which problems they used in their solutions. The number of points earned for each problem varies, but the total is 25 points. To receive full points, solutions must be clear mathematical proofs.

Continued Fractions

A number of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_N}}}}$$

is a *continued fraction* where the a_i are integers. We write this fraction as $[a_0, a_1, a_2, \ldots, a_N]$. We call $a_1, \ldots, a_N$ the *partial quotients* of the continued fraction.

We call $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$ the nth convergent.

To compute the continued fraction expansion of α, let $x_0 = \alpha$. Then let $a_i = \lfloor x_i \rfloor$ and $x_{i+1} = \frac{1}{x_i - a_i}$. We halt this process once $x_i = a_i$.

We can also deal with infinite convergents. $[a_0, a_1, a_2, \ldots]$. You may assume that all infinite continued fractions will converge. A continued fraction is periodic if $a_{l+k} = a_l$ for some fixed k and $l \geq L$ for some L. If the repeating part is $a_l, a_{l+1}, \ldots, a_n$, then we may write the continued fraction as $[a_0, a_1, \ldots, a_{l-1}, \overline{a_l, a_{l+1}, \ldots, a_n}]$.

1. (2 points.)

a. (0.5 points.) Evaluate $1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{3}}}}$

b. (0.5 points.) Convert into a continued fraction: $\frac{43}{30}$.

c. (0.5 points.) Find the continued fraction expansion of $\sqrt{3}$.

d. (0.5 points.) Evaluate $[2, \overline{2,4}]$.

Theorem 1. *If $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$, then for $n \geq 2$.*

$$\begin{aligned} p_0 &= a_0, & p_1 &= a_1 a_0 + 1, & p_n &= a_n p_{n-1} + p_{n-2} \\ q_0 &= 1, & q_1 &= a_1, & q_n &= a_n q_{n-1} + q_{n-2} \end{aligned}$$

2. (2 points.) Prove that

   a. (1 point.) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ for $n \geq 1$.

   b. (1 point.) $p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$ for $n \geq 2$.

3. (1 point.) Prove that nth convergent of $\frac{1+\sqrt{5}}{2}$ is equal to $\frac{F_n}{F_{n-1}}$ where F_n is the nth Fibonacci number and $F_0 = F_1 = 1$. (Note that $\frac{p_n}{q_n} = [a_0, \ldots, a_n]$ is the $(n+1)$th convergent since we started from 0. (Hint: What is the continued fraction expansion?)

4. (1 point.) Find a integer solution (x, y) to $1234x + 567y = 2$.

Theorem 2. *The even convergents to α are strictly increasing and are less than α while the odd convergents to α are strictly decreasing and are greater than α. In other words*

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \ldots < \alpha < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

5. (4 points.) Suppose α is irrational.

   a. (1 point.) Prove that $|\alpha - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}$. (Hint: What is the relationship of α with $\frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}$?)

   b. (2 points.) Of any two consecutive convergents to x, at least one satisfies the inequality $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$.

   c. (1 point.) Approximate $\sqrt{2}$ with a fraction with error $< \frac{1}{1000}$.

A *quadratic irrational* α is a number of the form $\frac{P+\sqrt{D}}{Q}$ with P, Q, D integers, $D > 0$ and D is not a perfect square. A quadratic irrational is *reduced* if $\alpha > 1$ and $-1 < \alpha' = \frac{P-\sqrt{D}}{Q} < 0$.

6. (6 points.) Suppose that α is a reduced quadratic irrational.

a. (1 point.) Suppose $\alpha = \frac{P+\sqrt{D}}{Q}$. Prove that $0 < P < \sqrt{D}$ and $0 < Q < 2\sqrt{D}$. (The converse isn't necessarily true but you don't have to prove this).

b. (2 points.) Show that if $\alpha = \lfloor \alpha \rfloor + \frac{1}{\alpha_1}$, then α_1 is a reduced quadratic irrational. You may assume that $(a+b)' = a' + b'$ and $(ab)' = a'b'$ where if $a = x + y\sqrt{D}$, then $a' = x - y\sqrt{D}$.

c. (3 points.) Show that if $\alpha_0 = \alpha$, $\alpha_i = q_i + \frac{1}{\alpha_{i+1}}$, then the sequence of α_i must be periodic where q_i is the floor of α_i. Periodic meaning that you can find N and l such that for $i > N$, $\alpha_{i+l} = \alpha_i$.

Theorem 3. *Any reduced quadratic irrational can be written as* $[\overline{a_0, a_1, \ldots, a_n}]$.

7. (2 points.)

a. (1 point.) Find, with proof, an integer m such $\sqrt{N} + m$ is reduced.

b. (1 point.) Conclude that the continued fraction expansion for $\sqrt{N}$ is of the form $[a_0, \overline{a_1, a_2, \ldots, a_n, 2a_0}]$.

Theorem 4. *Pick's Theorem: Area of convex polygon with vertices on lattice points is* $I + \frac{B}{2} - 1$ *where* I *is number of interior lattice points and* B *is number of boundary lattice points.*

Theorem 5. *Area of parallelogram spanned by* (a, b) *and* (c, d) *is the absolute value of the determinant of* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

For the next question, each part can be helpful in proving (c). Since we do not want to stifle mathematical creativity, if you can come with an alternate proof for (c), full credit for the entire question will still be given.

8. (7 points.) Let $\frac{P_i}{Q_i}$ be convergents to an irrational number $\alpha > 0$.

a. (2 points.) Prove that for each $n > 0$, there are no lattice points in the interior of the triangle whose vertices are $(0, 0), (P_n, Q_n), (P_{n+2}, Q_{n+2})$

b. (2.5 points.) Prove that if $|\alpha - \frac{P}{Q}| < \frac{1}{2Q^2}$, then there are no lattice points in interior of paralleologram spanned by $(2Q\alpha, 2Q)$ and $(2P - 2Q\alpha, 0)$ other than (P, Q). Note that the other point of the parellologram is $(2P, 2Q)$. Here P, Q are relatively prime. (Hint: If (x, y) is in the interior, what other point must also be).

c. (2.5 points.) Conclude that if $|\alpha - \frac{P}{Q}| < \frac{1}{2Q^2}$, then $\frac{P}{Q}$ is a convergent to α. You may assume that there exists n such that $\frac{P}{Q}$ is between $\frac{P_n}{Q_n}$ and $\frac{P_{n+2}}{Q_{n+2}}$ inclusively.