Duke Math Meet 2014

Team Round

1. Steven has just learned about polynomials and he is struggling with the following problem: expand $(1-2x)^7$ as $a_0 + a_1x + ... + a_7x^7$. Help Steven solve this problem by telling him what $a_1 + a_2 + \cdots + a_7$ is.

2. Each element of the set $\{2, 3, 4, ..., 100\}$ is colored. A number has the same color as any divisor of it. What is the maximum number of colors?

3. Fuchsia is selecting 24 balls out of 3 boxes. One box contains blue balls, one red balls and one yellow balls. They each have a hundred balls. It is required that she takes at least one ball from each box and that the numbers of balls selected from each box are distinct. In how many ways can she select the 24 balls?

4. Find the perfect square that can be written in the form $\overline{abcd} - \overline{dcba}$ where a, b, c, d are non zero digits and $b < c$. $\overline{abcd}$ is the number in base 10 with digits a, b, c, d written in this order.

5. Steven has 100 boxes labeled from 1 to 100. Every box contains at most 10 balls. The number of balls in boxes labeled with consecutive numbers differ by 1. The boxes labeled 1,4,7,10,...,100 have a total of 301 balls. What is the maximum number of balls Steven can have?

6. In acute $\triangle ABC$, AB=4. Let D be the point on BC such that $\angle BAD = \angle CAD$. Let AD intersect the circumcircle of $\triangle ABC$ at X. Let Γ be the circle through D and X that is tangent to AB at P. If $AP = 6$, compute AC.

7. Consider a 15×15 square decomposed into unit squares. Conisder a coloring of the vertices of the unit squares into two colors, red and blue such that there are 133 red vertices. Out of these 133, two vertices are vertices of the big square and 32 of them are located on the sides of the big square. The sides of the unit squares are colored into three colors. If both endpoints of a side are colored red then the side is colored red. If both endpoints of a side are colored blue then the side is colored blue. Otherwise the side is colored green. If we have 196 green sides, how many blue sides do we have?

8. Carl has 10 piles of rocks, each pile with a different number of rocks. He notices that he can redistribute the rocks in any pile to the other 9 piles to make the other 9 piles have the same number of rocks. What is the minimum number of rocks in the biggest pile?

9. Suppose that Tony picks a random integer between 1 and 6 inclusive such that the probability that he picks a number is directly proportional to the the number itself. Danny picks a number between 1 and 7 inclusive using the same rule as Tony. What is the probability that Tony's number is greater than Danny's number?

10. Mike wrote on the board the numbers $1, 2, ..., n$. At every step, he chooses two of these numbers, deletes them and replaces them with the least prime factor of their sum. He does this until he is left with the number 101 on the board. What is the minimum value of n for which this is possible?