Duke Math Meet 2012

Tiebreaker Round Solutions

1. An 8-inch by 11-inch sheet of paper is laid flat so that the top and bottom edges are 8 inches long. The paper is then folded so that the top left corner touches the right edge. What is the minimum possible length of the fold?

   Solution. Label the vertices of the rectangle clockwise from upper-left as A, B, C, D, so that A gets folded to $X \in CD$. Then the fold is the perpendicular bisector ℓ of AX. Then we have three cases: either ℓ intersects AB and CD, ℓ intersects AB and AD, or ℓ intersects AD and BC.

   In the first case let ℓ intersect AB at P and CD at Q. Write $\angle BAX = \theta$. Then construct $P' \in AB$ such that $P'D \parallel PQ$. Then we have $P'D = PQ$. Furthermore, triangle ADP' is similar to triangle BAX. Hence we have $PQ = P'D = AD \sec\theta$. This is minimized when $\theta = 0°$, or when A is folded to B and $PQ = 11$.

   In the second case, let ℓ intersect AB at P and AD at Q. Let PQ intersect AX at R, which is the midpoint of AX. Let H be the foot of the perpendicular from B to AX. Then we know that triangle AQP is similar to triangle BAX. Hence $PQ/AR = AX/BH$. But as $\angle ABX = 90°$, we may calculate the area of triangle ABX in two different ways to get $AX \cdot BH = AB \cdot BX$. Hence we have

   $$PQ = \frac{AR \cdot AX}{BH} = \frac{AX^2}{2BH} = \frac{AX^3}{2AB \cdot BX}.$$

   Writing $\theta = \angle BAX$, we find that $PQ = AB/(2\sin\theta\cos^2\theta)$; hence we need only maximize $\sin\theta\cos^2\theta$. Writing $u = \sin\theta$, we have $\sin\theta\cos^2\theta = \sin\theta - \sin^3\theta = u - u^3$. Making the substitution $u = 2\sin\phi/\sqrt{3}$, we have $u - u^3 = 2\sin(3\phi)/3\sqrt{3} \leq 2/3\sqrt{3}$. Hence we find that $PQ \geq 3\sqrt{3}AB/4 = 6\sqrt{3}$. (This minimization may also be done more quickly with calculus, but this particular non-calculus-based technique is rather nice.)

   In the third case, let ℓ intersect BC at P and AD at Q. Construct $Q' \in AD$ such that $BQ' \parallel PQ$. Then we know that $BQ' = PQ$. We know also that triangle $AQ'B$ is similar to triangle BAX, so that $BQ'/AB = AX/BX$. Hence to minimize $BQ' = PQ$, we need only minimize the ratio AX/BX. Writing $\theta = \angle BAX$, we have $AX/BX = \csc\theta$. To minimize $\csc\theta$, we want $\angle BAX$ as large as possible, so we take $X = C$. Then this gives $AX = \sqrt{8^2 + 11^2} = \sqrt{185}$ and $BX = 11$. Thus we get $PQ \geq 8\sqrt{185}/11$.

   In order to determine which case gives us the global minimum, we need to determine the relative ordering of $8\sqrt{185}/11$, $6\sqrt{3}$, and 11. As it turns out, we have $8\sqrt{185}/11 < 6\sqrt{3} < 11$, so the minimum possible length of the crease is $8\sqrt{185}/11$.

2. Triangle ABC is equilateral, with $AB = 6$. There are points D, E on segment AB (in the order A, D, E, B), points F, G on segment BC (in the order B, F, G, C), and points H, I on segment CA (in the order C, H, I, A) such that $DE = FG = HI = 2$. Considering all such configurations of D, E, F, G, H, I, let A_1 be the maximum possible area of (possibly degenerate) hexagon $DEFGHI$ and let A_2 be the minimum possible area. Find $A_1 - A_2$.

Solution. We know that $[DEFGHI] = [ABC] - ([ADI] + [BFE] + [CHG])$. Hence maximizing $[ADI] + [BFE] + [CHG]$ is equivalent to minimizing $[DEFGHI]$. Write $u = AD, v = BF, w = CH$. Then we know that

$$[ADI] + [BFE] + [CHG] = \frac{\sqrt{3}}{4}\left[u(4-w) + v(4-u) + w(4-v)\right],$$

where we have $0 \leq u, v, w \leq 4$. Clearly the minimum occurs when $u = v = w = 0$, so that $[DEFGHI] = [ABC] = 9\sqrt{3}$.

For convenience write $f(u, v, w) = u(4-w) + v(4-u) + w(4-v)$. Now we claim that $f(u, v, w) \leq 16$. We show that for any $0 \leq u, v, w \leq 4$, either $f(0, v, w) \geq f(u, v, w)$ or $f(4, v, w) \geq f(u, v, w)$. Indeed we have

$$\begin{aligned} f(0, v, w) - f(u, v, w) &= u(v + w - 4); \\ f(4, v, w) - f(u, v, w) &= (u-4)(v + w - 4). \end{aligned}$$

As u and $u - 4$ have opposite signs it follows that one of the two differences will be non-negative. Hence in maximizing f we may assume that $u, v, w \in \{0, 4\}$. To obtain a maximum clearly we cannot have $u = v = w = 0$ or $u = v = w = 4$. But if one or two of u, v, w are 4 and the others are 0, then $f(u, v, w) = 16$. Hence $f(u, v, w) \leq 16$ for all $0 \leq u, v, w \leq 4$.

Thus $[DEFGHI]$ achieves its minimum when $[ADI] + [BFE] + [CHG] = 4\sqrt{3}$, so that $[DEFGHI] = 9\sqrt{3} - 4\sqrt{3} = 5\sqrt{3}$. We find $A_1 - A_2 = 4\sqrt{3}$.

3. Find

$$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}.$$

Solution. By De Moivre's formula, we know that for $\theta \in \mathbb{R}$ we have

$$(\cos\theta + i\sin\theta)^k = \cos(n\theta) + i\sin(n\theta).$$

Take $k = 7$, $\theta = n\pi/7$, and consider the imaginary parts of both sides. The imaginary part of the right-hand side is zero, while we can find the imaginary part of the left-hand side by the binomial theorem. This gives

$$7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta = 0.$$

Dividing by $\sin^7\theta$, which is nonzero, gives

$$7\cot^6\theta - 35\cot^4\theta + 21\cot^2\theta - 1 = 0,$$

which holds for $\theta = \pi/7$, $\theta = 2\pi/7$, and $\theta = 3\pi/7$. Thus we have by Vieta's formulas that

$$\cot^2\frac{\pi}{7}\cot^2\frac{2\pi}{7}\cot^2\frac{3\pi}{7} = \frac{1}{7},$$

so inverting and taking square roots gives

$$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7} = \sqrt{7}.$$