Duke Math Meet 2011: Devil Round

In the Devil Round, all teams will be broken apart and re-assembled randomly to form new teams of about 10 people each. There will be 5 sets of 3 problems each for a total of 15 problems. At the start of the Devil Round, one person from each team will run to the front of the room and grab the first set of problems and run back to solve them with their team. When the team wishes to submit the answers to a set, one person from the team will run to the front of the room to submit their answers and grab the next set of problems to take back to their team. There will be a combined total time of 15 minutes to solve all 15 problems; awards will be given to the fastest team that obtains the highest score.

Devil Round Problems 1, 2, and 3

1. The fractal T-shirt for this year's Duke Math Meet is so complicated that the printer broke trying to print it. Thus, we devised a method for manually assembling each shirt — starting with the full-size 'base' shirt, we paste a smaller shirt on top of it. And then we paste an even smaller shirt on top of that one. And so on, infinitely many times. (As you can imagine, it took a while to make all the shirts.) The completed T-shirt consists of the original 'base' shirt along with all of the shirts we pasted onto it. Now suppose the base shirt requires 2011 cm^2 of fabric to make, and that each pasted-on shirt requires $4/5$ as much fabric as the previous one did. How many cm^2 of fabric in total are required to make one complete shirt?
2. A dog is allowed to roam a yard while attached to a 60-meter leash. The leash is anchored to a 40-meter by 20-meter rectangular house at the midpoint of one of the long sides of the house. What is the total area of the yard that the dog can roam?
3. 10 birds are chirping on a telephone wire. Bird 1 chirps once per second, bird 2 chirps once every 2 seconds, and so on through bird 10, which chirps every 10 seconds. At time $t = 0$, each bird chirps. Define $f(t)$ to be the number of birds that chirp during the t^{th} second. What is the smallest $t > 0$ such that $f(t)$ and $f(t + 1)$ are both at least 4?

Devil Round Problems 4, 5, and 6

4. The answer to this problem is 3 times the answer to problem 5 minus 4 times the answer to problem 6 plus 1.
5. The answer to this problem is the answer to problem 4 minus 4 times the answer to problem 6 minus 1.
6. The answer to this problem is the answer to problem 4 minus 2 times the answer to problem 5.

Devil Round Problems 7, 8, and 9

7. Vivek and Daniel are playing a game. The game ends when one person wins 5 rounds. The probability that either wins the first round is $1/2$. In each subsequent round the players have a probability of winning equal to the fraction of games that the player has lost. What is the probability that Vivek wins in six rounds?
8. What is the coefficient of x^8y^7 in $(1 + x^2 - 3xy + y^2)^{17}$?
9. Let $U(k)$ be the set of complex numbers z such that $z^k = 1$. How many distinct elements are in the union of $U(1), U(2), \cdots, U(10)$?

Devil Round Problems 10, 11, and 12

10. Evaluate $29\binom{30}{0}+28\binom{30}{1}+27\binom{30}{2}+...+0\binom{30}{29}-\binom{30}{30}$. You may leave your answer in exponential format.
11. What is the number of strings consisting of $2a$s, $3b$s and $4c$s such that a is not immediately followed by b, b is not immediately followed by c and c is not immediately followed by a?
12. Compute $(\sqrt{3}+\tan(1^\circ))(\sqrt{3}+\tan(2^\circ))\cdots(\sqrt{3}+\tan(29^\circ))$.

Devil Round Problems 13, 14, and 15

13. Three massless legs are randomly nailed to the perimeter of a massive circular wooden table with uniform density. What is the probability that the table will not fall over when it is set on its legs?
14. Compute
$$\sum_{n=1}^{2011} \frac{n+4}{n(n+1)(n+2)(n+3)}.$$
15. Find a polynomial in two variables with integer coefficients whose range is the positive real numbers.