DUKE MATH MEET 2009: TEAM ROUND

In the Team Round the entire team of six students will have 20 minutes to answer the 10 problems. The team members may collaborate freely, but as with all rounds in the Duke Math Meet, only pencil and paper may be used. After 20 minutes the team will submit a single sheet with its answers to each of the 10 problems. The team will be given 10-minute, 1-minute, and 15-second warnings. Each correct answer will add 2 points to the team's score.

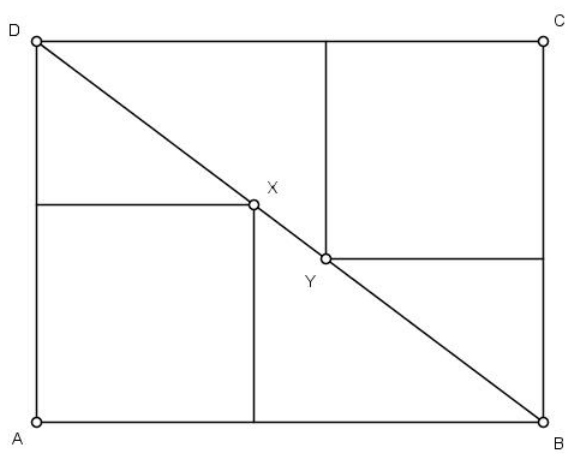
Team Round

1. You are on a flat planet. There are 100 cities at points $x = 1, \dots, 100$ along the line $y = -1$, and another 100 cities at points $x = 1, \dots, 100$ along the line $y = 1$. The planet's terrain is scalding hot, and you cannot walk over it directly. Instead, you must cross archways from city to city. There are archways between all pairs of cities with different y coordinates, but no other pairs: for instance, there is an archway from $(1, -1)$ to $(50, 1)$, but not from $(1, -1)$ to $(50, -1)$. The amount of "effort" necessary to cross an archway equals the square of the distance between the cities it connects. You are at $(1, -1)$, and you want to get to $(100, -1)$. What is the least amount of effort this journey can take?

2. Let $f(x) = x^4 + ax^3 + bx^2 + cx + 25$. Suppose a, b, c are integers and $f(x)$ has 4 distinct integer roots. Find $f(3)$.

3. Frankenstein starts at the point $(0, 0, 0)$ and walks to the point $(3, 3, 3)$. At each step he walks either one unit in the positive x-direction, one unit in the positive y-direction, or one unit in the positive z-direction. How many distinct paths can Frankenstein take to reach his destination?

4. Let $ABCD$ be a rectangle with $AB = 20$, $BC = 15$. Let X and Y be on the diagonal $\overline{BD}$ of $ABCD$ such that $BX > BY$. Suppose A and X are two vertices of a square which has two sides on lines $\overline{AB}$ and $\overline{AD}$, and suppose that C and Y are vertices of a square which has sides on $\overline{CB}$ and $\overline{CD}$. Find the length XY.



5. $n \geq 2$ kids are trick-or-treating. They enter a haunted house in a single-file line such that each kid is friends with precisely the kids (or kid) adjacent to him. Inside the haunted house, they get mixed up and out of order. They meet up again at the exit, and leave in single file. After leaving, they realize that each kid (except the first to leave) is friends with at least one kid who left before him. In how many possible orders could they have left the haunted house?

6. Call a set S *sparse* if every pair of distinct elements of S differ by more than 1. Find the number of sparse subsets (possibly empty) of $\{1, 2, \dots, 10\}$.

7. How many ordered triples of integers (a, b, c) are there such that $1 \leq a, b, c \leq 70$ and $a^2 + b^2 + c^2$ is divisible by 28?

8. Let C_1, C_2 be circles with centers O_1, O_2, respectively. Line ℓ is an external tangent to C_1 and C_2; it touches C_1 at A and C_2 at B. Line segment $\overline{O_1O_2}$ meets C_1 at X. Let C be the circle through A, X, B with center O. Let $\overline{OO_1}$ and $\overline{OO_2}$ intersect circle C at D and E, respectively. Suppose the radii of C_1 and C_2 are 16 and 9, respectively, and suppose the area of the quadrilateral O_1O_2BA is 300. Find the length of segment $\overline{DE}$.

9. What is the remainder when $5^{5^{5^5}}$ is divided by 13?

10. Let α and β be the smallest and largest real numbers satisfying

$$x^2 = 13 + \lfloor x \rfloor + \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{4} \right\rfloor.$$

Find $\beta - \alpha$. ($\lfloor a \rfloor$ is defined as the largest integer that is not larger than a.)