DUKE MATH MEET 2008: TEAM ROUND

In the Team Round the entire team of six students will have 20 minutes to answer the 10 problems. The team members may collaborate freely, but like all rounds in the Duke Math Meet only pencil and paper can be used. After 20 minutes the team will submit a single sheet with their answers to each of the 10 problems. The team will be given 10-minute, 1-minute, and 15-second warnings. Each correct answer will add 2 points to their team's score.

Team Round

1. $ABCD$ is a convex quadrilateral such that $AB = 20$, $BC = 24$, $CD = 7$, $DA = 15$, and $\angle DAB$ is a right angle. What is the area of $ABCD$?

2. A triangular number is one that can be written in the form $1+2+\cdots+n$ for some positive number n. 1 is clearly both triangular and square. What is the next largest number that is both triangular and square?

3. Find the last (i.e. rightmost) three digits of 9^{2008}.

4. When expressing numbers in a base $b \geq 11$, you use letters to represent digits greater than 9. For example, A represents 10 and B represents 11, so that the number 110 in base 10 is $A0$ in base 11. What is the smallest positive integer that has four digits when written in base 10, has at least one letter in its base 12 representation, and no letters in its base 16 representation?

5. A fly starts from the point $(0, 16)$, then flies straight to the point $(8, 0)$, then straight to the point $(0, -4)$, then straight to the point $(-2, 0)$, and so on, spiraling to the origin, each time intersecting the coordinate axes at a point half as far from the origin as its previous intercept. If the fly flies at a constant speed of 2 units per second, how many seconds will it take the fly to reach the origin?

6. A line segment is divided into two unequal lengths so that the ratio of the length of the short part to the length of the long part is the same as the ratio of the length of the long part to the length of the whole line segment. Let D be this ratio. Compute:

$$D^{-1} + D^{[D^{-1}+D^{(D^{-1}+D^2)}]}.$$

7. Let $f(x) = 4x + 2$. Find the ordered pair of integers (P, Q) such that their greatest common divisor is 1, P is positive, and for any two real numbers a and b, the sentence:

"$Pa + Qb \geq 0$"

is true if and only if the following sentence is true:

"For all real numbers x, if $|f(x) - 6| < b$, then $|x - 1| < a$."

8. Call a rectangle "simple" if all four of its vertices have integers as both of their coordinates and has one vertex at the origin. How many simple rectangles are there whose area is less than or equal to 6?

9. A square is divided into eight congruent triangles by the diagonals and the perpendicular bisectors of its sides. How many ways are there to color the triangles red and blue if two ways that are reflections or rotations of each other are considered the same?

10. In chess, a knight can move by jumping to any square whose center is $\sqrt{5}$ units away from the center of the square that it is currently on. For example, a knight on the square marked by the horse in the diagram below can move to any of the squares marked with an "X" and to no other squares. How many ways can a knight on the square marked by the horse in the diagram move to the square with a circle in exactly four moves?