DUKE MATH MEET 2007: TEAM ROUND

In the Team Round the entire team of six students will have 20 minutes to answer the 10 problems. The team members may collaborate freely, but like all rounds in the Duke Math Meet only pencil and paper can be used. After 20 minutes the team will submit a sheet with their answers to each of the 10 problems. The team will be given 10-minute, 1-minute, and 15-second warnings. Each correct answer will add 2 points to their team's score.

Team Round

1. If $x + z = v$, $w + z = 2v$, $z - w = 2y$, and $y \neq 0$, compute the value of
$$\left(x + y + \frac{x}{y}\right)^{101}.$$
2. Every minute, a snail picks one cardinal direction (either north, south, east, or west) with equal probability and moves one inch in that direction. What is the probability that after four minutes the snail is more than three inches away from where it started?
3. What is the probability that a point chosen randomly from the interior of a cube is closer to the cube's center than it is to any of the cube's eight vertices?
4. Let $ABCD$ be a rectangle where $AB = 4$ and $BC = 3$. Inscribe circles within triangles ABC and ACD. What is the distance between the centers of these two circles?
5. C is a circle centered at the origin that is tangent to the line $x - y\sqrt{3} = 4$. Find the radius of C.
6. I have a fair 100-sided die that has the numbers 1 through 100 on its sides. What is the probability that if I roll this die three times that the number on the first roll will be greater than or equal to the sum of the two numbers on the second and third rolls?
7. List all solutions (x, y, z) of the following system of equations with x, y, and z positive real numbers:
$$\begin{aligned} x^2 + y^2 &= 16 \\ x^2 + z^2 &= 4 + xz \\ y^2 + z^2 &= 4 + yz\sqrt{3}. \end{aligned}$$
8. $A_1A_2A_3A_4A_5A_6A_7$ is a regular heptagon (7 sided-figure) centered at the origin where $A_1 = \left(\sqrt[91]{6}, 0\right)$. $B_1B_2B_3 \cdots B_{13}$ is a regular triskaidecagon (13 sided-figure) centered at the origin where $B_1 = \left(0, \sqrt[91]{41}\right)$. Compute the product of all lengths A_iB_j, where i ranges between 1 and 7, inclusive, and j ranges between 1 and 13, inclusive.
9. How many three-digit integers are there such that one digit of the integer is exactly two times a digit of the integer that is in a different place than the first? (For example, 100, 122, and 124 should be included in the count, but 42 and 130 should not.)
10. Let α and β be the solutions of the quadratic equation:
$$x^2 - 1154x + 1 = 0.$$
Find $\sqrt[4]{\alpha} + \sqrt[4]{\beta}$.