DUKE MATH MEET 2007: POWER ROUND: INTEGER-VALUED POLYNOMIALS

In the Power Round the entire team of six students will have 60 minutes to answer this series of proof-based questions. The team members may collaborate freely, but like all rounds in the Duke Math Meet only pencil and paper can be used. After 60 minutes the team will submit all solutions. Teams will be given 30-minute, 5-minute, and 1-minute warnings. Teams may use results of previous problems to solve later problems, even if the team has not submitted solutions to those previous problems. The number of points earned for each problem varies, but the total is 16 points. This power round is divided into three somewhat independent parts.

Part 1: All Integer Linear Combinations of Binomial Coefficients are Integer-Valued Polynomials: Total: 5 Points

This power round is concerned with integer-valued polynomials, in other words polynomials $p(x)$ such that for every integer z, $p(z)$ is an integer. There are many polynomials with this property; an obvious class of such polynomials is those polynomials with integer cofficients, for example, $3x^2$, $x-1$, $x^{17}+17$, or $-4x^2+5x-3$. But there are more integer-valued polynomials than these; a polynomial $p(x)$ with rational coefficients such that $p(z)$ is an integer for every integer z is $\frac{1}{2}x^2 - \frac{1}{2}x = \frac{x(x-1)}{2}$. This is because if z is an integer, then one of z and $z-1$ will be even, so $z(z-1)$ will be even, and therefore $\frac{z(z-1)}{2}$ will be an integer. However, not every polynomial $p(x)$ with rational coefficients is integer-valued. For example, the polynomial $q(x) = \frac{x}{2}$ is a polynomial with rational coefficients but is not integer valued, since $q(1) = \frac{1}{2} \notin \mathbb{Z}$.

1. (0.5 points) However, notice that $q(z) = \frac{z}{2}$ is an integer for some integers z, for example, $q(2) = \frac{2}{2} = 1$. Give an example of a simple, nonconstant polynomial $r(x)$ with rational coefficients such that $r(z)$ is never an integer for any integer z. You do not need to explain why $r(z)$ is never an integer for any integer z, you only need to give the polynomial.

Now, the polynomial $\frac{x(x-1)}{2}$ is an example of what is called a binomial coefficient polynomial. For positive integers n, we can define the nth binomial coefficient polynomial $\binom{x}{n}$ as:

$$\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-(n-2))(x-(n-1))}{n(n-1)(n-2)\cdot\cdots\cdot 2\cdot 1},$$

and we also define $\binom{x}{0}$ to be 1. For example, $\binom{x}{1} = \frac{x}{1} = x$, $\binom{x}{2} = \frac{x(x-1)}{2\cdot 1} = \frac{1}{2}x^2 - \frac{1}{2}x$, and $\binom{x}{3} = \frac{x(x-1)(x-2)}{3\cdot 2\cdot 1} = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{3}x$. A fact that may be surprising is that any binomial coefficient polynomial is an integer-valued polynomial. You will now prove this fact in two different ways:

2. (1 point) Explain why, for any two nonnegative integers z and n, where z and n are not both 0, $\binom{z}{n}$ is the number of ways of choosing a subcommittee of n people from a committee of z people, where the order of choosing does not matter.
3. (1 point) Explain why the previous result shows that $\binom{x}{n}$ is an integer-valued polynomial for any nonnegative integer n.
4. (2 points) Let n be any positive integer, and let p be any positive prime number less than or equal to n. Let x_1, x_2, x_3, $\ldots$, x_n be any n consecutive integers. Give an expression (not necessarily closed form) for the the largest power of p that evenly divides $n(n-1)\cdot\cdots\cdot 2\cdot 1$ and show by factoring $x_1x_2\ldots x_n$ that this power of p evenly divides $x_1x_2\ldots x_n$.
5. (0.5 points) Explain why the previous result shows that $\binom{x}{n}$ is an integer-valued polynomial for any positive integer n.
6. (0.5 points) A polynomial $p(x)$ is said to be an integer linear combination of binomial coefficients if it can be written in the form:

$$a_n\binom{x}{n} + a_{n-1}\binom{x}{n-1} + \cdots + a_1\binom{x}{1} + a_0\binom{x}{0},$$

where a_n, a_{n-1}, $\ldots$, a_1, a_0 are all integers. For example, x^2 is an integer linear combination of binomial coefficients because:

$$2\binom{x}{2} + 1\binom{x}{1} = (x^2 - x) + x = x^2.$$

Using previous results, explain why any integer linear combination of binomial coefficients is an integer-valued polynomial.

Part 2: Any Integer-Valued Polynomial is an Integer Linear Combination of Binomial Coefficients: Total: 6 Points

As the title of this section implies, we will be proving that the integer linear combinations of binomial coefficients give all integer-valued polynomials. The key tool to this analysis is the shift operator, which is written as Δ. For any function f, we can define a new function Δf as:

$$(\Delta f)(x) = f(x+1) - f(x).$$

For example, if $f(x) = x + 2$, then $(\Delta f)(x) = f(x+1) - f(x) = (x+3) - (x+2) = 1$.

7. (0.5 points) Compute and fully expand the polynomials Δx^2 and Δx^3. You do not need to show any computations to receive full credit; simply give the results.
8. (0.5 points) Show that for any functions f and g and any constants u and v, $\Delta(uf+vg) = u\Delta f + v\Delta g$.
9. (0.5 points) Show that if f is a polynomial then Δf is also a polynomial, and if f is an integer-valued polynomial then Δf is also an integer-valued polynomial.
10. (0.5 points) Show that if f is a polynomial and Δf is the zero polynomial, then f must be a constant polynomial. Further, explain why if f is integer-valued, then the constant polynomial f must be an integer.
11. (0.5 points) Using previous results, show that if f and g are polynomials and $\Delta f = \Delta g$, then $f = g + C$ for some constant C. Also show that if f and g are also integer-valued, then C is an integer.
12. (1 point) Let $f(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$ be a general polynomial of degree n, where n is a positive integer and the leading term c_n is nonzero. Prove that the degree of Δf is $n-1$ and its leading term is nc_n.
13. (0.5 points) Show that $\Delta\binom{x}{n+1} = \binom{x}{n}$ for all nonnegative integers n.

The technique of mathematical induction is a very common technique mathematicians use to prove statements of the form "for all positive integers n, ..." or "for all nonnegative integers n, ..." Its main idea is as follows: in order to prove that a fact holds for all nonnegative integers n, all one needs to do is: (1) show that the fact is true if $n = 0$ and (2) show that for any nonnegative integer k, if the fact is true if $n = k$, then the fact is true if $n = k + 1$. Induction works because we have shown that the fact is true if $n = 0$, and since the fact is true for $n = 0$ the fact is also true for $n = 0 + 1 = 1$, and since the fact is true for $n = 1$ the fact is also true for $n = 1 + 1 = 2$, and one can work one's way up to any nonnegative integer. We will now show using induction and the previous results that any integer-valued polynomial is an integer linear combination of binomial coefficients.

14. (0.5 points) Show that if f is an integer-valued polynomial and the degree of f is 0 (i.e. f is a constant function), then f is an integer linear combination of binomial coefficients.
15. (1 point) Show that if for some nonnegative integer k all integer-valued polynomials of degree k are integer linear combination of binomial coefficients and f is any integer-valued polynomial of degree $k+1$, then f is an integer linear combination of binomial coefficients.
16. (0.5 points) Explain why the previous two facts allow us to conclude that any integer-valued polynomial f is an integer linear combination of binomial coefficients from induction on the degree of f.

Part 3: Loose Ends: Total: 5 Points

17. (0.5 points) Part of the result of problem 10 is that if f is a polynomial and $\Delta f = 0$, then f must be a constant. However, there are non-polynomial functions (and therefore non-constant functions) g such that $\Delta g = 0$. Give a simple example of such a function.
18. (1 point) Although you (hopefully) showed that any integer-valued polynomial is an integer linear combination of binomial coefficients, you still don't know if there are integer-valued polynomials that can be written in more than one way as an integer linear combination of binomial coefficients. In other words, you don't yet know if representations of integer-valued polynomials as integer linear combinations of binomial coefficients are unique. Show that these representations are unique, in other words, if for some integer-valued polynomial $f(x)$ there are nonnegative integers m and n and integers a_0, a_1, $\ldots$, a_m, b_0, b_1, $\ldots$, b_n such that:
$$\begin{aligned} f(x) &= a_m\binom{x}{m} + a_{m-1}\binom{x}{m-1} + \cdots + a_1\binom{x}{1} + a_0\binom{x}{0} \\ &= b_n\binom{x}{n} + b_{n-1}\binom{x}{n-1} + \cdots + b_1\binom{x}{1} + b_0\binom{x}{0}, \end{aligned}$$
then $m = n$ and $a_0 = b_0$, $a_1 = b_1$, $\ldots$, $a_m = b_m = b_n$.
19. (1.5 points) We also never found a simple way of, for a given integer-valued polynomial $f(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0$, computing what the nonnegative integer m and the integers a_0, a_1, $\ldots$, a_{m-1}, a_m are such that:
$$f(x) = a_m\binom{x}{m} + a_{m-1}\binom{x}{m-1} + \cdots + a_1\binom{x}{1} + a_0\binom{x}{0}.$$
We also have not shown if there is a computationally simple way of determining whether a polynomial is integer-valued. (We have shown that a polynomial is integer-valued if and only if it is an integer linear combination of binomial coefficients, but we have not given a computationally useful way of showing if a polynomial can be written as an integer linear combination of binomial coefficients.) Give an algorithm that will solve both problems at once and prove it works. More specifically, give an algorithm that can determine whether a given polynomial is integer-valued and if it is, determine what the nonnegative integer m and integers a_0, $\ldots$, a_m are such that
$$f(x) = a_m\binom{x}{m} + a_{m-1}\binom{x}{m-1} + \cdots + a_1\binom{x}{1} + a_0\binom{x}{0},$$
and explain why it works.
20. (0.5 points each = 2 points total) Using the algorithm from the previous question or some other method, determine whether each of the following polynomials are integer-valued, and if they are integer-valued, write it as an integer linear combination of binomial coefficients. (You do not have to give any computations; you may just state the results.)
    (a) $\frac{(x-1)x(x+1)(x+2)}{5\cdot 4\cdot 3\cdot 2\cdot 1}$.
    (b) $\frac{(x-2)(x-1)x(x+1)(x+2)}{5\cdot 4\cdot 3\cdot 2\cdot 1}$.
    (c) $\frac{1}{15}x^5 - \frac{1}{2}x^4 + \frac{11}{6}x^3 + \frac{8}{5}x$.
    (d) x^5.