GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE ENERGY-CRITICAL NONLINEAR SCHRÖDINGER EQUATION IN $\mathbb{R}^3$

In this talk I will present some recent work in collaboration with J. Colliander, M. Keel, H. Takaoka and T. Tao.

We consider the Cauchy problem for the quintic defocusing Schrödinger equation in $\mathbb{R}^{1+3}$

$$\begin{cases} iu_t + \Delta u = |u|^4 u \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

where $u(t,x)$ is a complex-valued field in spacetime $\mathbb{R}_t \times \mathbb{R}^3_x$. This equation has as Hamiltonian,

$$E(u(t)) := \int \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6 \; dx. \tag{2}$$

Since the Hamiltonian (2) is preserved by the flow (1) we shall often refer to it as the *energy* and write $E(u)$ for $E(u(t))$.

Semilinear Schrödinger equations - with and without potentials, and with various nonlinearities - arise as models for diverse physical phenomena, including Bose-Einstein condensates and as a description of the envelope dynamics of a general dispersive wave in a weakly nonlinear medium. Our interest here in the defocusing quintic equation is motivated mainly though by the fact that the problem is critical with respect to the energy norm. Specifically, we map a solution to another solution through the scaling $u \mapsto u^\lambda$ defined by

$$u^\lambda(t,x) := \frac{1}{\lambda^{1/2}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}), \tag{3}$$

and this scaling leaves both terms in the energy invariant.

The Cauchy problem for the equation (1) has been intensively studied, just to name few authors we recall here Bourgain, Grillakis, Ginibre and Velo, Kato. Cazenave and Weissler proved that if the initial data $u_0(x)$ has finite energy, then the Cauchy problem is locally well-posed, in the sense that there exists a local-in-time solution to which lies in $C^0_t \dot{H}^1_x \cap L^{10}_{t,x}$, and is unique in this class; furthermore the map from initial data to solution is locally Lipschitz continuous in these norms. If the energy is small, then the solution is known to exist globally in time, and scatters to a solution to the free Schrödinger equation. In the presence of large initial data, the arguments of Cazenave and Weissler do not extend to yield global well-posedness, even though for this equation the energy is conserved. This is because the time of existence given by the local theory depends on the profile of the data as well as on the energy[1].

For large finite energy data which is assumed to be in addition radially symmetric, Bourgain proved global existence and scattering for the equation (1) in

[1]This is in contrast with sub-critical equations such as the cubic equation, for which one can use the local well-posedness theory to yield global well-posedness and scattering even for large energy data.

$\dot{H}^1(\mathbb{R}^3)$. Subsequently Grillakis gave a different argument which recovered part of Bourgain's result - namely, global existence from smooth, radial, finite energy data-. For general large data - in particular, general smooth data - global existence and scattering were open.

Our main result is the following global well-posedness result for (1) in the energy class.

Theorem 1. *For any u_0 with finite energy, $E(u_0) < \infty$, there exists a unique*[2] *global solution $u \in C^0_t(\dot{H}^1_x) \cap L^{10}_{t,x}$ to (1) such that*

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} |u(t,x)|^{10} \, dxdt \le C(E(u_0)). \tag{4}$$

for some constant $C(E(u_0))$ that depends only on the energy.

As is well-known for the sub-critical analogue), the $L^{10}_{t,x}$ bound above also gives scattering, asymptotic completeness, and uniform regularity. In particular our result shows that classical solutions for the equation (1) exist!

The method of proof that we follow is similar in spirit to the induction-on-energy strategy of Bourgain , but we perform the induction analysis in both frequency space and physical space simultaneously, and replace the Morawetz inequality by an interaction variant. The principal advantage of the interaction Morawetz estimate is that it is not localized to the spatial origin and so is better able to handle nonradial solutions. In particular, this interaction estimate, together with an almost-conservation argument controlling the movement of L^2 mass in frequency space, rules out the possibility of energy concentration.

[2] In fact, uniqueness actually holds in the larger space $C^0_t(\dot{H}^1_x)$ (thus eliminating the constraint that $u \in L^{10}_{t,x}$), as one can show by adapting the arguments of.