COMBINATORIAL POSITIVITY BY GEOMETRIC DEGENERATION

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Why do so many polynomials that arise naturally in various branches of mathematics and physics have positive integer coefficients when expressed in terms of simpler polynomials? For a quintessential example, consider the ubiquitous Schur polynomials, which expand positively as sums of monomials. Among the many other important examples are the *Schubert polynomials* of Lascoux and Schützenberger, which also expand positively into monomials, and the *Stanley symmetric functions*, which expand positively as sums of Schur functions. One of the main goals of algebraic combinatorics is to prove the positivity of expansions into sums by exhibiting sets of combinatorial objects counted by the coefficients. For Schur polynomials, these objects are the *semistandard Young tableaux*, while the coefficients of Schubert polynomials count the more general *reduced pipe dreams* (also known as *rc-graphs*) of Fomin and Kirillov. This talk will be about certain polynomials recently discovered by Buch and Fulton, which arise in the context of representation theory of quivers. The combinatorial objects in this case are certain line diagrams that have a similar feel to those above.

Polynomials that interest the combinatorics community have a habit of (already or eventually) being just as interesting to geometers, who naturally desire an explanation of the positivity from their own point of view. Frequently the geometry involves orbits of Lie group actions, with the polynomials appearing as topological invariants. The Schur and Schubert polynomials enter this way, for instance, in the context of Grassmann and flag manifolds, which carry actions of general linear groups. Historically, geometric perspectives have lent extra depth and meaning to positive combinatorial formulas. But occasionally, as in the case of Buch–Fulton quiver polynomials to be described in this talk, the geometry surrounding Lie group actions actually arrives at the combinatorics (and the positivity) before the combinatorialists.

Commutative algebra jumps into the fray as a language to connect geometry with combinatorics. More precisely, Gröbner bases describe the algebra governing certain degenerations of varieties. The limits ('special fibers') of these degenerations usually break the original variety into multiple pieces, and the algebra allows one to get a concrete handle on the combinatorics of these pieces, which can be quite subtle. For Schur, Schubert, and Buch–Fulton polynomials, the algebra concerns minors in matrices filled with variables, and Gröbner bases replace these complicated equations with the squarefree products of variables that comprise their "antidiagonal" terms.

Given that our desired polynomial is expressed as a topological invariant of some algebraic variety, and that this invariant doesn't change in well-behaved families of varieties, we find the *same* topological invariant is assigned to the degenerate special fiber, which has multiple components. Each component gets assigned its own topological invariant, and a positive sum then results by adding the contributions from the components.

The purpose of the talk will be to introduce the geometric, algebraic, and combinatorial viewpoints, as well as their interactions, by focusing on the example of Buch–Fulton quiver polynomials. The exposition will be completely elementary: I will assume no familiarity with Schur and Schubert polynomials, tableaux and pipe dreams, Grassmann and flag varieties, or Gröbner bases. For audience members who nevertheless want to do some background reading, I suggest looking at the first 29 pages of [KM03], as much of [BF99] as possible, and the first 20 pages of [KMS03]. It might also be helpful to glance through [Man01] and Chapters 14–17 of [MS03], the latter of which (for the purposes here) is distilled from the above references as well as prior developments in [BB93, BJS93, FK96, FS94, Ful92, LM98, LS82, Sta84, Zel85].

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