SOME p-ADIC DIFFERENTIAL EQUATIONS IN ARITHMETIC GEOMETRY

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Since Dwork's pioneering work in the sixties, it has been known at some level that there are deep connections between

- complex (algebraic and analytic) differential equations,
- p-adic (algebraic and analytic) differential equations, and
- numbers of points on (and zeta functions of) algebraic varieties over finite fields.

I would like to survey two facets of this (rather long) story, one quite old, one quite new.

The old: Say X is a family of smooth proper varieties over some base B, all defined over $\mathbb{Z}$ and having good reduction modulo p (i.e., everything is still smooth when you reduce the equations modulo p; for given varieties, this is true for all but finitely many p). Then the algebraic de Rham cohomology (i.e., the hypercohomology of the sheaf of algebraic/meromorphic differential forms) of the fibres of the family form a local system on B, where the connection is the Gauss-Manin connection (or in more classical language, the associated Picard-Fuchs equation). That is, they form a vector bundle on B equipped with an integrable connection.

The goal of this section will be to expose what was just said in the previous paragraph concretely in a situation considered by Dwork: the Legendre family of elliptic curves

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

over the affine line with coordinate λ (minus 0 and 1). In this case the local system (of rank 2) boils down to a classical hypergeometric differential equation (with parameters $1/2, 1/2, 1$), and for λ in a finite field $\mathbb{F}_q$ with $q = p^n$, you can compute the number of points on E_λ by "evaluating" a certain ratio of hypergeometric functions at the Teichmüller lift of λ. (The quotes are because the point of evaluation is outside the p-adic disc where the individual hypergeometrics converge, but their ratio converges on a larger region.) This ends up happening because there is a "Frobenius structure" on the Gauss-Manin connection which computes numbers of points.

Our discussion of this will be a summary of [9, Section 7], which includes a fairly explicit development of this example in the context of a p-adic cohomology theory (Monsky-Washnitzer cohomology).

Aside (not to be discussed in the talk): nowadays, this sort of setup is used for machine computations of zeta functions, most notably for hyperelliptic curves (where the computations are quite similar to the above case). In particular, [4] considers a single fibre at a time, while [6] uses the Gauss-Manin connection to exhibit the Frobenius structure as the solution of a p-adic differential equation with initial condition.

The new: The new wrinkle is to try to understand versions of the above story *without* assuming they come from a specific situation in algebraic geometry. This is loosely akin to certain aspects of the Langlands program. Compare for instance Wiles's work attaching a modular form to a rational elliptic curve; the modular form "forgets" the defining equation of the elliptic curve and keeps track only of some cohomology.

The aspect I want to focus on here is *monodromy*, which in the classical setting refers to the way local solutions of a differential equation (i.e., horizontal sections of a local system) are moved around by parallel transport along a noncontractible loop. Unfortunately, one doesn't have a literal analogue of parallel transport in p-adic geometry, because the topology is much too disconnected. However, much progress has been made lately in getting around this.

I'll focus on a key local situation, which I first describe in the complex setting. There is an old result of Borel [8, Lemma 4.5] to the effect that given a local system on a punctured open complex disc equipped with a polarized variation of Hodge structure (a family of lattices in the vector bundle, a direct sum decomposition and a bilinear form satisfying certain compatibilities), the monodromy of the local system is quasi-unipotent (i.e., its eigenvalues are roots of unity). Thus after adjoining a certain root of the parameter of the disc, you can find a nilpotent basis of sections, i.e., one horizontal section, a second section which becomes horizontal modulo the first section, et cetera. The main examples of such local systems are those coming from (primitive) relative holomorphic de Rham cohomology of a family of smooth proper analytic varieties over the punctured disc; the variation of Hodge structure is the integral cohomology modulo torsion plus the Hodge decomposition.

Now pass to the p-adic setting, where we consider an open unit disc over $\mathbb{Q}_p$ with a closed disc of smaller radius removed. (For technical reasons, removing a point is not enough in many applications.) One can make sense of "analytic geometry" over $\mathbb{Q}_p$ (via Tate's theory of rigid analytic geometry), but for our purposes it's enough to simply know what the "analytic functions" are on our p-adic annulus: if its radii are $\epsilon < 1$, the functions are

$$\{\sum_{n=-\infty}^{\infty} c_n t^n : c_n \in K, \lim_{n\to\pm\infty} |c_n|\rho^n = 0 \qquad \text{for } \epsilon < \rho < 1\}.$$

Now say I have a "local system", i.e., a finite projective module over this power series ring equipped with a connection (automatically integrable because we're in dimension 1). While one doesn't have parallel transport with which to define monodromy, one can perfectly well talk about a nilpotent basis of sections, and ask whether one can find such a basis after pulling back by some cover of the annulus by another annulus. (These covers are parametrized by finite extensions of a power series field in characteristic p, so they can be *much* more complicated than just taking a root of a parameter!) Of course, one needs some replacement for the Hodge-theoretic condition; it is the existence of a "Frobenius structure" of the sort considered in the first part.

The goal of this section is to give a precise statement of the resulting analogue of Borel's theorem, which was formulated conjecturally by Crew and Tsuzuki and recently established independently by André [1], Mebkhout [7], and myself [5]. (The first two proofs rely heavily on results of Christol and Mebkhout [3].) I might also mention an example or two (due originally to Dwork) where the monodromy has been worked out explicitly

I probably won't get to say anything about applications, but here are a few:

- The theorem plays a role analogous to that of Grothendieck's local monodromy theorem in étale cohomology; in particular, it is needed to establish the finite dimensionality of the cohomology of a "p-adic local system" (overconvergent F-isocrystal). The complex analytic theory is also converging towards this point of view; for instance, recent work of Bloch and Esnault associates to complex local systems invariants typically constructed from étale sheaves to complex local systems.

- Using Berthelot's construction of rigid cohomology (a Weil cohomology constructed using Dwork-type methods), the theorem can be used as part of a transcription of Deligne's second proof of the Weil conjectures (or rather, Laumon's Fourier-theoretic simplification of same) into p-adic language. The point is that one is proving the Weil conjectures using a cohomology theory defined explicitly in terms of defining equations of the original variety, rather than implicitly in terms of unramified covers (as with étale cohomology).
- Maybe the most surprising application comes from the work of Berger [2]. Building on work of Cherbonnier-Colmez (and ultimately Fontaine), he found a link between Fontaine's theory of p-adic Galois representations (i.e., continuous representations over the p-adic field $\mathbb{Q}_p$ of $\mathrm{Gal}(\overline{K}/K)$, for K a finite extension of $\mathbb{Q}_p$) and p-adic differential equations. This lets him describe, in terms of differential equations, much of Fontaine's classification of such representations into types (de Rham, crystalline, semistable, potentially crystalline, potentially semistable...) which appear to arise in the étale cohomology of various types of varieties over K. In particular, he established a conjecture of Fontaine that "de Rham representations are potentially semistable". (Each category is constructed as follows. Given your representations, you tensor over $\mathbb{Q}_p$ with some "period ring" on which Galois acts, take invariants, and keep only the representations which you can recover from the invariants.)

References

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