Finiteness theorems for quadratic forms

The classical "Four Squares Theorem" of Lagrange (1770) asserts that any positive integer can be expressed as the sum of four squares—that is, the quadratic form $x^2 + y^2 + z^2 + t^2$ represents all (positive) integers. When does a general (positive definite) quadratic form represent all (positive) integers? This question was first posed, and addressed in a systematic way, by Ramanujan in his classic 1916 paper [10]. In this paper, Ramanujan discovered and wrote down 54 more quaternary (four-variable) quadratic forms representing all integers! His list was as follows:

$$\begin{array}{lllllll}
[1,1,1,1], & [1,1,1,2], & [1,1,1,3], & [1,1,1,4], & [1,1,1,5], & [1,1,1,6], & [1,1,1,7],\\
[1,1,2,2], & [1,1,2,3], & [1,1,2,4], & [1,1,2,5], & [1,1,2,6], & [1,1,2,7], & [1,1,2,8],\\
[1,1,2,9], & [1,1,2,10], & [1,1,2,11], & [1,1,2,12], & [1,1,2,13], & [1,1,2,14], & [1,1,3,3],\\
[1,1,3,4], & [1,1,3,5], & [1,1,3,6], & [1,2,2,2], & [1,2,2,3], & [1,2,2,4], & [1,2,2,5],\\
[1,2,2,6], & [1,2,2,7], & [1,2,3,3], & [1,2,3,4], & [1,2,3,5], & [1,2,3,6], & [1,2,3,7],\\
[1,2,3,8], & [1,2,3,9], & [1,2,3,10], & [1,2,4,4], & [1,2,4,5], & [1,2,4,6], & [1,2,4,7],\\
[1,2,4,8], & [1,2,4,9], & [1,2,4,10], & [1,2,4,11], & [1,2,4,12], & [1,2,4,13], & [1,2,4,14],\\
[1,2,5,6], & [1,2,5,7], & [1,2,5,8], & [1,2,5,9], & [1,2,5,10] & &
\end{array}$$

where we have used $[a, b, c, d]$ as shorthand for the diagonal form $ax^2 + by^2 + cz^2 + dt^2$. In fact, the above list was later shown to contain every *diagonal* quaternary quadratic form representing all integers. ("diagonal" means having no cross terms)

Ramanujan's article was the starting point for a huge amount of work on the subject by numerous authors. There were papers proving Ramanujan's assertions—a project completed by Dickson [4]—as well as papers attempting to extend Ramanujan's list, such as those by Pall and by Halmos [6]. The most notable accomplishment on the latter topic was the 1950 Ph.D. thesis of Willerding [13], who found, with proof, 168 quaternary integer-matrix forms representing all integers. (An *integer-matrix* quadratic form is one whose cross-terms all have even coefficients, or, equivalently, whose corresponding matrix has integer entries.) At the time, Willerding's list was thought to be complete.

Surprisingly, the real progress on the question of enumerating *universal* forms—i.e., forms representing all positive integers—did not come until very recently, when Conway and Schneeberger (1993) proved a very remarkable theorem. They showed that: an integer-matrix positive definite quadratic form represents all positive integers if and only if it represents all positive integers up to 15.

In this talk, we will describe a simple proof of Conway-Schneeberger's "Fifteen Theorem", and we will also discuss how the theorem generalizes, leading to numerous such "finiteness theorems for quadratic forms". In particular, we will consider the questions: When does a quadratic form represent all odd integers? When does it represent all primes? Both these questions turn out to have some very surprising answers.

Finally, we will state and consider the ultimate variant of Conway-Schneeberger's Fifteen Theorem, namely their fundamental "290 conjecture".

The main ingredients in the proofs will be the language of lattices, the notion of genus, and (for the generalizations) the basic language of theta series and modular forms. For a beautiful introduction to these subjects, see Conway's little book [3], and also Conway's and Hanke's recent survey articles [2] and [7]. The other references given below are also themed around the topics we will discuss.

References

[1] M. Bhargava, On the Conway-Schneeberger Fifteen Theorem, *Quadratic Forms and their Applications* (Dublin), Contemp. Math. **272**, Amer. Math. Soc., Providence, RI (1999), 27–37.

[2] J. H. Conway, Universal Quadratic Forms and the Fifteen Theorem, *Quadratic Forms and their Applications* (Dublin), Contemp. Math. **272**, Amer. Math. Soc., Providence, RI (1999), 23–26.

[3] J. H. Conway, *The sensual (quadratic) form*, Carus Mathematical Monographs **26**, Mathematical Association of America, Washington, DC, 1997.

[4] L. E. Dickson, Quaternary quadratic forms representing all integers, *Amer. J. of Math.* **49** (1927), 35–56.

[5] W. Duke and R. Schulze-Pillot, Representation of integers by positive ternary quadratic forms and equidistribution of lattice points on ellipsoids. *Invent. Math.* **99** (1990), no. 1, 49–57.

[6] P. R. Halmos, *Note on almost-universal forms*, *Bull. Amer. Math. Soc.* **33** (1927), 141–144.

[7] J. Hanke, Some recent results about (ternary) quadratic forms, http://www.math.princeton.edu/ jonhanke/Web-O2/Ternary-QF-Survey/Ternary-QF-Survey.pdf .

[8] B. Jones and G. Pall, Regular and semi-regular positive ternary quadratic forms, *Acta. Math.* **70** (1939), 165–191.

[9] H. D. Kloosterman, On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$, *Acta Math.* **49** (1926), 407–464.

[10] S. Ramanujan, On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$, *Proc. Camb. Phil. Soc.* **19** (1916), 11–21.

[11] W. A. Schneeberger, *Arithmetic and Geometry of Integral Lattices*, Ph.D Thesis, Princeton University, 1995.

[12] G. L. Watson, One-class genera of positive ternary quadratic forms-II, *Mathematika* **22**, no. 43, 1–11.

[13] M. F. Willerding, Determination of all classes of positive quaternary quadratic forms which represent all (positive) integers, *Bulletin of the American Mathematical Society* **54** (1948), no. 3, 334–337.