ABSTRACT – Allen Knutson

Some recent collisions of combinatorics with algebraic geometry

Given a k-dimensional subspace V of $\mathbb{C}^n$ and a full flag of subspaces $\mathcal{A} = \big(0 = A_0 < A_1 < A_2 < \ldots < A_n = \mathbb{C}^n\big)$, we can ask where the dimensions of intersections jump; $\big(\dim(V \cap A_i)/(V \cap A_{i-1})\big)$ will be a bit-string of k ones and $n-k$ zeroes. In a similar way, the possible ways for two flags to intersect (up to linear automorphism of the ambient vector space) are indexed by permutations of n elements. In these ways (and of course, many others) combinatorics enters algebraic geometry.

Several questions in algebraic geometry have recently seen progress by combinatorial methods:

1. *Positive Schubert calculus on Grassmannians*
Given three generically situated flags $\mathcal{A}, B, C$ in $\mathbb{C}^n$, for which triples of intersection conditions do there exist subspaces V satisfying all three conditions?

2. As stated above, there are only discretely many ways for two flags to sit relative to one another, up to linear automorphism. To think about the moduli of *triples* of flags, we need to use geometric invariant theory, which requires a choice of ample line bundle. For which such line bundles is this geometric invariant theory quotient nonempty? This is also a GL_n-representation theory problem.

(This has a more easily stated symplectic geometry counterpart – given three Hermitian matrices with sum zero, what conditions must their eigenvalues satisfy?)

3. Let F be the manifold of full flags in $\mathbb{C}^n$, and $\mathcal{A}$ a fixed flag. On F we have n tautological vector bundles, and by taking successive quotients we get a list of line bundles, whose first Chern classes generate the cohomology (Borel's theorem). Given a permutation w (and thereby an intersection condition), let X_w be the "Schubert variety" in F consisting of those flags intersecting $\mathcal{A}$ "w-much." How can one write the Poincaré dual of X_w as a polynomial in those Chern classes?

Note that while a solution to Problem 3 can be made to give a formula to count the number of V's in Problem 1, this formula is an alternating sum, and is not much use for determining positivity.

Problem 1 turns out to be related to Problem 2 in two very different ways, allowing for a recursive solution to both! While this recursion is quite clear on the combinatorial side, it remains rather mysterious in the algebraic geometry. No clear role has yet been seen here for the much harder problem of Schubert calculus on flag manifolds.

Problem 3 was given a nice solution in the early '90s by combinatorialists, in a way that shows that the Schubert polynomials have positive coefficients: they count "RC-graphs". This has recently been given a couple of algebro-geometric interpretations, one being that the closely related matrix Schubert varieties have a Gröbner degeneration to a union of subspaces indexed by RC-graphs. In this case, the algebraic geometry gives back to combinatorics – the proof of the Gröbner basis property gives a long-sought algorithm for generating RC-graphs inductively.

Literature

1. Bergeron, Nantel; Billey, Sara. *RC-graphs and Schubert polynomials.* Experiment. Math. 2 (1993), no. 4, 257–269.

2. Fomin, Sergey; Kirillov, Anatol N. *The Yang-Baxter equation, symmetric functions, and Schubert polynomials.* Discrete Math. 153 (1996), no. 1-3, 123–143.

3. Knutson, Allen; Tao, Terence. *The honeycomb model of* $\mathrm{GL}_n(C)$ *tensor products.* I. Proof of the saturation conjecture. J. Amer. Math. Soc. 12 (1999), no. 4, 1055–1090.

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6. Java applet: http://www.math.ucla.edu/~tao/java/Honeycomb.html