ABSTRACT – Brian Conrad

Modular Curves and rigid analytic spaces

Most geometers believe that one cannot really do "geometry" over a non-archimedean complete field, in contrast to the situation over the complex field. Tate and others developed the theory of rigid analytic geometry in order to at least make coherent sheaf theory (including $GAGA$) work nicely over such totally disconnected fields, but the spaces involved only barely qualified as "geometric" objects: when working with such spaces one has to deal with a variety of unpleasant technical problems. Rigid analytic methods led to deep results in the study of abelian varieties and other situations of number-theoretic interest, but one could not really define (etale) cohomology for such spaces and it was all probably still viewed as a bit esoteric by those in other fields. In more recent years, Berkovich has developed an enhancement of rigid geometry in which the underlying topological spaces are very nice (e.g., locally compact and locally path connected!), and this has completely transformed the subject. Perhaps the most spectacular example is the prominent role of Berkovich's theory as the geometric foundation for the Harris-Taylor proof of the local Langlands conjecture for $GL(n)$.

By considering a relatively concrete geometric question about modular curves, we will see the attraction of the "classical" theory of Tate and also how this theory has some serious geometric deficiencies which are magically eliminated by adopting Berkovich's foundations instead. The motivation for the geometric question arises from work of Katz in the early 1970's which showed that the arithmetic of modular forms could be investigated by working with modular curves over p-adic fields, viewed as rigid analytic spaces and not as algebraic curves. This point of view has remained of vital interest. The moduli space property of modular curves as both algebraic and complex-analytic objects is what makes them so useful in the study of modular forms, so a natural question (which could be avoided in Katz' work) is this: are the rigid-analytic incarnations of modular curves also moduli spaces, in the appropriate geometric category? The answer is "yes", but the verification of this is by no means trivial; in particular, the method used in the complex-analytic case (exponential uniformization) is not applicable.

No previous exposure to the theory of modular curves or rigid analytic spaces will be assumed. It is hoped that the existence of both genuine moduli spaces and a category of p-adic spaces with nice topological properties will provide compelling evidence that geometry over a p-adic field is not at all esoteric (though some experience is required in order to get used to it).

References

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